MATHEMATICS MAGAZINE

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VOLUME 44 · JANUARY 1971 · NUMBER 1

MATHEMATICS MAGAZINE

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ADVERTISING CORRESPONDENCE should be addressed to RAOUL HAILPERN, Mathematical Association of America, SUNY at Buffalo (University of Buffalo), Buffalo, New York 14214.

The MATHEMATICS MAGAZINE is published by the Mathematical Association of America at Washington D. C., bi-monthly except July–August. Ordinary subscriptions are: 1 year $4.00. Members of the Mathematical Association of America and of Mu Alpha Theta may subscribe at the special rate of 2 years for $6.00. Single copies are 80¢.

Second class postage paid at Washington, D. C. and additional mailing offices.

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MATHEMATICS MAGAZINE

Published by

THE MATHEMATICAL ASSOCIATION OF AMERICA

(INCORPORATED)

A JOURNAL OF COLLEGIATE MATHEMATICS

VOLUME 44

1971

Published at

MENASHA, WIS., AND WASHINGTON, D. C.

STATEMENT OF POLICY

The MATHEMATICS MAGAZINE is published to promote and advance college mathematics and is intended in particular to be interesting and useful to junior college and community college mathematics teachers. It publishes a wide variety of mathematical material of interest as well to students, to teachers at both the college and secondary school level, and to people outside the academic community who are interested in mathematics. It is aimed at a mathematically less sophisticated audience than that of the MONTHLY. We consider its principal function to be helping its readers maintain an active interest in mathematics and helping them extend their knowledge and deepen their understanding of mathematics, its applications, its history, and the problems involved in its teaching.

We believe it important to insist upon the highest possible standards of exposition in the articles published. We want them to be read and to be understood. For publication in this MAGAZINE, an article should not require a high level of technical knowledge of mathematics. It should be interesting and, in some reasonable sense, it should be relevant both to the current mathematical scene and to the undergraduate mathematics program. We shall welcome manuscripts satisfying these conditions. In the *Book Reviews* section books and films of special interest to our readers will be reviewed. *Problems and Solutions* is a crucial and essential part of this MAGAZINE; we solicit problems of a wide variety and a broad range of levels of difficulty.

Finally we welcome criticism and suggestions for improvement.

G. N. WOLLAN

A NOTE ON INTEGRATION

CASPER GOFFMAN, Purdue University

1. Introduction. In his treatment of a Riemann type integral for functions on the real line, N. Bourbaki, [1], features the class of regulated functions. These functions are the limits of uniformly convergent sequences of step functions. They may be characterized as those functions whose right and left sided limits exist everywhere.

This class of functions, while rather attractive in the one variable case, seems to be quite uninteresting for functions of more than one variable. In the first place, the class is not coordinate invariant. Secondly, no such nice characterization as the one mentioned above for regulated functions on the line holds in higher dimensions. If step functions are defined in terms of simplicial decompositions the coordinate invariance is restored but the functions still have no interesting continuity structure.

The purpose here is to note that a small change in the kind of convergence considered results in an integral which is suitable in all dimensions. Indeed, it yields the Riemann integral itself and, with a small twist, the Lebesgue integral.

Consider r dimensional space with a fixed rectangular coordinate system $(x_1, \cdots, x_r)$. Let $I = [a_1, b_1] \times \cdots \times [a_r, b_r]$ be a closed interval. We consider partitions $\pi = \{I_1, \cdots, I_m\}$ of I into finite sets of nonoverlapping closed intervals. This means that $\bigcup_{k=1}^{m} I_k = I$ and that $i \neq j$ implies the intersection of the interiors of I_i and I_j is empty, although I_i and I_j may have boundary points in common.

By the content of an interval we mean its r dimensional volume. An elementary set S is one which is the union of a finite set of nonoverlapping intervals; the sum of the contents of these intervals is the content of S.

We define a distance for real functions f and g on I. For each partition π of I and each $k > 0$, let $\pi(k; f, g)$ consist of those intervals $J \in \pi$ such that $|f(x) - g(x)| > k$ for some $x \in J$. Let $E(f, g)$ be the set of positive numbers k such that, for some partition π, the content of $\pi(k; f, g)$ does not exceed k, and let

$$d(f, g) = \inf E(f, g).$$

Accordingly $d(f, g) < k$ if and only if there is a partition of I such that if $I_j, j = 1, \cdots, r$, are the intervals of the partition on each of which

$$\max[|f(x) - g(x)| : x \in I_j] < k,$$

the content of $I \sim \bigcup_{j=1}^{r} I_j$ is less than k.

2. The Riemann integral. Let f be a bounded real function on I. A step function g is one which is constant on the interior of each interval of a partition $\pi = \{I_1, \cdots, I_m\}$ of I. If $g(x) = c_i$ for all x in the interior of I_i, $i = 1, \cdots, m$, we define the integral of g as $\int_I g = \sum_{i=1}^{m} c_i |I_i|$, where $|I_i|$ is the content of I_i. The bounded function f is said to be integrable if there is a uniformly bounded sequence $\{g_n\}$ of step functions such that $\lim_n d(f, g_n) = 0$.

(a) If f is bounded and $\{g_n\}$ is a uniformly bounded sequence of step functions with $\lim_n d(f, g_n) = 0$, then $\lim_n \int_I g_n$ exists.

Proof. There is an M such that $|f(x)| \leqq M$ and $|g_n(x)| \leqq M$ for each $x \in I$ and $n = 1, 2, \cdots$. Let $\epsilon > 0$. There is an N such that $m, n > N$ implies $d(g_m, g_n) < \epsilon$. Then $|\int_I g_m - \int_I g_n| < \epsilon |I| + 2\epsilon M$. This proves that the sequence of integrals converges.

(b) If f is bounded on I and $\{g_n\}$ and $\{h_n\}$ are uniformly bounded sequences of step functions with $\lim_n d(f, g_n) = 0$ and $\lim_n d(f, h_n) = 0$ then $\lim_n \int_I g_n = \lim_n \int_I h_n$.

Proof. The sequence $\{g_1, h_1, \cdots, g_n, h_n, \cdots\}$ satisfies (a).

Properties (a) and (b) allow us to define the integral of an integrable f as $\int_I f = \lim_n \int_I g_n$, where $\{g_n\}$ is any uniformly bounded sequence of step functions with $\lim_n d(f, g_n) = 0$.

(c) Suppose f is bounded and integrable. Let $k > 0$. There is a partition π

such that the elementary set, composed of those intervals of π in which the saltus of f, [2, p. 94], exceeds k, has content less than k. This implies that f is Riemann integrable.

For the converse, suppose f is Riemann integrable. Let $k>0$. The set of points at which the saltus of f, [2, p. 95] exceeds k can be covered by finitely many intervals the sum of whose contents is less than k. Let $M \geqq f(x)$ for all $x \in I$. There is a step function g such that $d(f, g) < k$ and $|g(x)| \leqq M+2k$ for all $x \in I$. It follows easily that f is integrable.

We have thus shown that our integral is the Riemann integral. We can obtain the main properties of the Riemann integral via our definition, but we shall not give the details.

3. Measurable functions. We define sets of measure 0. S has measure 0 if for every $\epsilon>0$ there is a sequence $\{I_n\}$ of intervals such that $S \subset \bigcup_{n=1}^{\infty} I_n$ and $\sum_{n=1}^{\infty} |I_n| < \epsilon$. If a property holds at each $x \in I$, except for a set of measure 0, the property is said to hold almost everywhere.

We consider Cauchy sequences of step functions in our metric. A sequence $\{g_n\}$ is Cauchy if for each $\epsilon>0$ there is an N such that $m, n > N$ implies $d(g_m, g_n) < \epsilon$. Let $\{g_n\}$ be a Cauchy sequence. Then $\{g_n\}$ has a subsequence $\{h_n\}$ such that, for each n and $m>n$, $d(h_m, h_n) < 1/2^n$. It follows that $\{h_n\}$ converges almost everywhere to a function h defined almost everywhere on I. Let $\{k_n\}$ be a second subsequence of $\{g_n\}$ which converges almost everywhere to a function f defined almost everywhere on I. By choosing an appropriate subsequence of $\{k_n\}$ it follows that f and h agree almost everywhere. Thus, every Cauchy sequence determines a function defined except for possible changes on sets of measure 0.

In order to see that all measurable functions are obtained in this way we must of course know some measure theory. We need here only the approximation of measurable sets by means of elementary sets, [2, p. 158], from which it follows that if f is measurable there is a sequence $\{g_n\}$ of step functions that converges almost everywhere to f. That $\{g_n\}$ is a Cauchy sequence in our sense follows from the fact that if g and h are step functions which differ by less than k except on a set of measure less than k then $d(g, h) < k$.

4. Summable functions. We describe the nonnegative summable functions in terms of special Cauchy sequences of step functions. We need the integral $\int_E g$ of a step function on an elementary set E. This is defined by $\int_E g = \int_I g_E$ where $g_E(x) = g(x)$, $x \in E$ and $g_E(x) = 0$, $x \notin E$.

LEMMA. *If $\{g_n\}$ is a nondecreasing sequence of nonnegative step functions such that $\{\int_I g_n\}$ is bounded then, for each $\epsilon>0$, there is a $\delta>0$ such that for each elementary set E, with $|E| < \delta$, we have $\int_E g_n < \epsilon$ for all $n = 1, 2, \cdots$.*

Proof. Let $\{g_n\}$ be a nondecreasing sequence of nonnegative step functions. Suppose there is a $k>0$ such that for any $\delta>0$ there is an elementary set E and an n such that $|E| < \delta$ and $\int_E g_n > k$. Then $\int_I g_n > k$.

Hence suppose that, for some nonnegative integer j, there is an n with

$\int_I g_n > kj$. For this fixed function g_n, there is a $\delta > 0$ such that, for any elementary set E with $|E| < \delta$, we have $\int_{I \sim E} g_n > kj$. Since, for some $m > n$, there is a set E with $|E| < \delta$ and $\int_E g_m > k$, we obtain

$$\int_I g_m = \int_{I \sim E} g_m + \int_E g_m \geqq \int_{I \sim E} g_n + \int_E g_m > k(j+1).$$

It follows that $\{\int_E g_m\}$ is unbounded and the lemma is proved.

Let $\{g_n\}$ be a Cauchy sequence of nonnegative step functions. For each n, let $h_n = \sup\{g_1, \cdots, g_n\}$. We say that the measurable f determined by $\{g_n\}$ is summable if the sequence $\{\int_I h_n\}$ is bounded.

If $\{g_n\}$ is a Cauchy sequence of nonnegative step functions such that $\{\int_I h_n\}$ is bounded then $\lim_n \int_I g_n$ exists.

Proof. For each n, let $h_n = \sup\{g_1, \cdots, g_n\}$. Then $\{h_n\}$ is a nondecreasing sequence of nonnegative step functions. Let $\epsilon > 0$. By the lemma there is a $\delta > 0$ such that if E is elementary and $|E| < \delta$ then $\int_E h_n < \epsilon$ for $n = 1, 2, \cdots$. There is an N such that $m, n > N$ implies $d(g_m, g_n) < \min(\epsilon, \delta)$. Then $|\int_I g_m - \int_I g_n| < \epsilon |I| + 2\epsilon$, proving the result.

The integral of f determined by $\{g_n\}$ is defined by $\int_I f = \lim_n \int_I g_n$.

This class agrees with the nonnegative summable functions in the usual sense. Let f be nonnegative and summable in the usual sense. Suppose $\int_I f > 0$, and let $f = \sum_{n=1}^{\infty} f_n$ where each f_n is nonnegative and $\int_I f_n > 0$. Let g_1 be a nonnegative step function such that $\int_I g_1 < \int_I f_1 + \frac{1}{2}$ and, if E_1 is the set for which $f_1(x) > g_1(x)$ then $\int_{E_1} f_1 < \frac{1}{2^2}$. Let g_2 be a nonnegative step function such that $\int_I (g_1 + g_2) < \int_I (f_1 + f_2) + \frac{1}{2} + \frac{1}{2^2}$ and if E_2 is the set for which $f_1(x) + f_2(x) > g_1(x) + g_2(x)$ then $\int_{E_2} (f_1 + f_2) < \frac{1}{2^3}$. Continuing in this way, we obtain a series $\sum_{n=1}^{\infty} g_n$ of nonnegative step functions such that $\sum_{n=1}^{\infty} g_n \geqq \sum_{n=1}^{\infty} f_n$ almost everywhere and, with $h_n = g_1 + \cdots + g_n$, $n = 1, 2, \cdots$, we have $\int_I h_n < \int_I f + 1$.

Now, let $\{u_n\}$ be a sequence of nonnegative step functions which is Cauchy in our metric and which converges almost everywhere to f. Let $v_n = \inf(u_n, h_n)$, $n = 1, 2, \cdots$. Then $\{v_n\}$ is a Cauchy sequence in our metric. It converges almost everywhere to f. The sequence of integrals $\int_I \sup(v_1, \cdots, v_n)$, $n = 1, 2, \cdots$, is bounded. Thus f is summable in our sense.

We may define summability for functions which are both positive and negative in the usual way. That the sum of two summable functions f and g is summable and that $\int_I (f+g) = \int_I f + \int_I g$ follows quickly, as do various other properties of the integral.

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2. C. Goffman, Real Functions, Prindle, Weber and Schmidt, Boston, 1953.

COVARIANT AND CONTRAVARIANT VECTORS

S. R. DEANS, University of South Florida, Tampa

1. Introduction. The subject of covariant and contravariant tensors commonly comes up in both undergraduate mathematics and physics courses. Although most seniors and graduate students have heard the terms "covariant" and "contravariant", it is the author's observation that very few of them have any mental picture of the distinction between these concepts. The purpose of this paper is to show a graphical distinction between covariant and contravariant vectors. The interpretation here is certainly not new [1], and it is easy to obtain by use of generalized coordinates [2]; however, the approach we use can be understood by both students and teachers with a minimum of mathematical training.

2. Two coordinate systems. In Figure 1, we have two coordinate systems. The x-system, with coordinate axes x^1 and x^2, is orthogonal. (Note that the 1 and 2 are superscripts and not powers.) The y-system, with coordinate axes y^1 and y^2, is not orthogonal. The two systems are related by a linear transformation which we shall display in Section 3. There is only one way to label the coordinates of the point P with respect to the x-system. The first coordinate is given by $\alpha^1 = OA$ and the second coordinate is $\alpha^2 = OB$. A different situation presents itself with respect to the y-system. The coordinates of P can be given by $(\beta^1, \beta^2) = (OE, OF)$ or $(\beta_1, \beta_2) = (OC, OD)$. We shall show later that β^1 and β^2 are contravariant components and β_1 and β_2 are covariant components. It is important to note that the contravariant components are found by drawing the appropriate parallel lines, $PF \| Oy^1$ and $PE \| Oy^2$, while the covariant components are found by dropping the appropriate perpendiculars, $PC \perp Oy^1$ and $PD \perp Oy^2$. Clearly, there is no distinction between covariant and contravariant components with respect to the x-system. Thus we have $(\alpha^1, \alpha^2) = (\alpha_1, \alpha_2) = (OA, OB)$. In accordance with tradition, we use superscripts on the contravariant components and subscripts on the covariant components.

3. Contravariant components. It is clear from Figure 1 that the contravariant components are given by (note that $\beta^2 = EP$)

(3.1a) $$\alpha^1 = OH + GP = \beta^1 \cos\theta + \beta^2 \sin\phi$$

and

(3.1b) $$\alpha^2 = HE + EG = \beta^1 \sin\theta + \beta^2 \cos\phi.$$

We solve this system for β^1 and β^2 to obtain

(3.2a) $$\beta^1 = \frac{\alpha^1 \cos\phi - \alpha^2 \sin\phi}{\cos(\theta + \phi)}$$

and

(3.2b) $$\beta^2 = \frac{-\alpha^1 \sin\theta + \alpha^2 \cos\theta}{\cos(\theta + \phi)}\,.$$

The coordinate transformation equations are of the same form as those given in (3.1),

$$x^1 = y^1 \cos\theta + y^2 \sin\phi \tag{3.3a}$$

and

$$x^2 = y^1 \sin\theta + y^2 \cos\phi. \tag{3.3b}$$

It is interesting to note that these equations can be written in the general form

$$x^1 = y^1 \cos(x^1, y^1) + y^2 \cos(x^1, y^2) \tag{3.4a}$$

and

$$x^2 = y^1 \cos(x^2, y^1) + y^2 \cos(x^2, y^2), \tag{3.4b}$$

where $\cos(x^i, y^j)$ means the cosine of the angle between the x^i-axis and the y^j-axis. These linear coordinate transformation equations can be generalized to n dimensions.

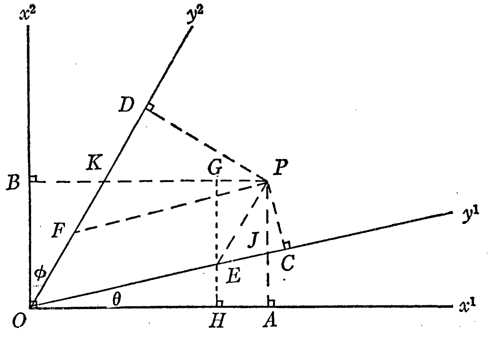

FIG. 1. The x-system and the y-system. The following relations hold. $PA \perp Ox^1$, $PB \perp Ox^2$, $PC \perp Oy^1$, $PD \perp Oy^2$, $PE \| Oy^2$, $PF \| Oy^1$, $GH \| PA$, $\angle GEP = \phi$, $\angle APC = \theta$, $\angle EPC = \theta + \phi$.

4. **Covariant components.** From Figure 1 we have

$$\begin{aligned} \beta^1 &= OC = OJ + JC = OJ + JP \sin\theta \\ &= \alpha_1 \sec\theta + (AP - AJ) \sin\theta \\ &= \alpha_1 \sec\theta + \alpha_2 \sin\theta - \alpha_1 \sin\theta \tan\theta \\ &= \alpha_1 \cos\theta + \alpha_2 \sin\theta \end{aligned} \tag{4.1a}$$

and

$$\begin{aligned} \beta_2 &= OD = OK + KD = OK + KP \sin\phi \\ &= \alpha_2 \sec\phi + (BP - BK) \sin\phi \\ &= \alpha_2 \sec\phi + \alpha_1 \sin\phi - \alpha_2 \tan\phi \sin\phi \\ &= \alpha_1 \sin\phi + \alpha_2 \cos\phi. \end{aligned} \tag{4.1b}$$

We solve these equations for α_1 and α_2 and obtain

$$\alpha_1 = \frac{\beta_1 \cos\phi - \beta_2 \sin\theta}{\cos(\theta + \phi)} \tag{4.2a}$$

and

$$\alpha_2 = \frac{-\beta_1 \sin\phi + \beta_2 \cos\theta}{\cos(\theta + \phi)}. \tag{4.2b}$$

We can now make use of the relation $\alpha^i = \alpha_i$ and (3.1) and (4.2) to obtain the relationship between the β_i and β^i,

$$\beta_1 = \beta^1 + \beta^2 \sin(\theta + \phi) \tag{4.3a}$$

and

$$\beta_2 = \beta^1 \sin(\theta + \phi) + \beta^2. \tag{4.3b}$$

5. Transformation properties. The transformation properties of the covariant and contravariant components of a vector are well defined [2]. We shall give these definitions for a two-dimensional space and then use them to show that our interpretation given above is consistent with the definitions.

In the following definitions the β's refer to the y-system, the α's refer to the x-system, and the coordinate transformation equations are given by a system of the form $x^i = x^i(y^1, y^2)$ or $y^i = y^i(x^1, x^2)$, where $(i = 1, 2)$. That is, we assume that the transformation equations can be solved for the x^i in terms of the y^i or they can be solved for the y^i in terms of the x^i.

DEFINITION 5.1. *A contravariant vector is the entire class of quantities such as* $\alpha^i(x)$, $\beta^i(y)$, $\cdots$, *related to one another by the transformation of the form*

$$\beta^i = \sum_{j=1}^{2} \frac{\partial y^i}{\partial x^j} \alpha^j \qquad (i = 1, 2). \tag{5.1}$$

DEFINITION 5.2. *A covariant vector is the entire class of quantities* $\alpha_i(x)$, $\beta_i(y)$, $\cdots$, *related to one another by the transformation of the form*

$$\beta_i = \sum_{j=1}^{2} \frac{\partial x^j}{\partial y^i} \alpha_j \qquad (i = 1, 2). \tag{5.2}$$

We now apply these definitions to the two coordinate systems shown in Figure 1. From (3.3) we can calculate the needed derivatives,

$$\frac{\partial x^j}{\partial y^i} = \begin{pmatrix} \cos\theta & \sin\phi \\ \sin\theta & \cos\phi \end{pmatrix}, \tag{5.3}$$

and after solving (3.3) for the y^i we find

$$\frac{\partial y^i}{\partial x^j} = \frac{1}{\cos(\theta + \phi)} \begin{pmatrix} \cos\phi & -\sin\phi \\ -\sin\theta & \cos\theta \end{pmatrix}. \tag{5.4}$$

The contravariant components are thus given by

$$\beta^1 = \frac{\partial y^1}{\partial x^1}\alpha^1 + \frac{\partial y^1}{\partial x^2}\alpha^2 = \frac{\alpha^1 \cos\phi - \alpha^2 \sin\phi}{\cos(\theta+\phi)}, \tag{5.5a}$$

and

$$\beta^2 = \frac{\partial y^2}{\partial x^1}\alpha^1 + \frac{\partial y^2}{\partial x^2}\alpha^2 = \frac{-\alpha^1 \sin\theta + \alpha^2 \cos\theta}{\cos(\theta+\phi)}, \tag{5.5b}$$

in agreement with (3.2). The covariant components are given by

$$\beta_1 = \frac{\partial x^1}{\partial y^1}\alpha_1 + \frac{\partial x^2}{\partial y^1}\alpha_2 = \alpha_1 \cos\theta + \alpha_2 \sin\theta \tag{5.6a}$$

and

$$\beta_2 = \frac{\partial x^1}{\partial y^2}\alpha_1 + \frac{\partial x^2}{\partial y^2}\alpha_2 = \alpha_1 \sin\phi + \alpha_2 \cos\phi, \tag{5.6b}$$

in agreement with (4.1).

The connection between the β^i and the β_i is given in terms of the metric tensor $g_{ij}(y)$ for the y-system. The defining equation for $g_{ij}(y)$ is [2] (in two dimensions)

$$g_{ij}(y) = \sum_{k=1}^{2} \frac{\partial x^k}{\partial y^i}\frac{\partial x^k}{\partial y^j}, \qquad (i, j = 1, 2). \tag{5.7}$$

If we expand (5.7) and use (5.3) we obtain the explicit form

$$g_{ij}(y) = \begin{pmatrix} 1 & \sin(\theta+\phi) \\ \sin(\theta+\phi) & 1 \end{pmatrix}. \tag{5.8}$$

The covariant and contravariant components are then related by [2]

$$\beta_i = \sum_{j=1}^{2} g_{ij}\beta^j \qquad (i = 1, 2) \tag{5.9}$$

or explicitly

$$\beta_1 = \beta^1 + \beta^2 \sin(\theta+\phi) \tag{5.10a}$$

and

$$\beta_2 = \beta^1 \sin(\theta+\phi) + \beta^2 \tag{5.10b}$$

in agreement with (4.3).

We have thus shown that the graphical interpretation of covariant and contravariant vectors is consistent with the definitions.

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AN APPLICATION OF THE OSCILLATION OF A FUNCTION AT A POINT

TOM C. VENABLE, JR., Indiana State University

The purpose of this note is to demonstrate an application of the oscillation of a function f at a point z_0.

DEFINITION. *$w(f, z_0)$ will denote the oscillation of f at the point z_0, i.e., $w(f, z_0) = \inf \sup |f(x) - f(y)|$ where the* sup *is taken over all points x and y in an open neighborhood containing z_0 and the* inf *is taken over all such open neighborhoods.*

It is well known that f is continuous at z_0 if and only if $w(f, z_0) = 0$. This concept will be used to obtain information about the set of continuous functions on an interval $[a, b]$; this set will be denoted $C[a, b]$. Let $B[a, b]$ be the set of all bounded functions over $[a, b]$. Using the oscillation concept, it will be shown that $C[a, b]$ is a closed subset of $B[a, b]$ in the uniform topology. A more elementary proof uses the triangle inequality and the fact that a function which is continuous on $[a, b]$ is uniformly continuous on $[a, b]$. [1].

We will show that $B[a, b] - C[a, b]$ is open. Note that this proof uses the direct definition of an open set in a metric space, i.e., a set is open if and only if it is a neighborhood of each of its points. Let $f \in B[a, b] - C[a, b]$. One must show that there exists some sphere about f completely contained in $B[a, b] - C[a, b]$. Since f is not continuous over $[a, b]$, there exists some point $z_0 \in [a, b]$ such that $w(f, z_0) = \delta > 0$. Consider the open sphere about f of radius $\delta/4$, i.e., $\{g \mid \sup|f(x) - g(x)| < \delta/4\}$ where x ranges over $[a, b]$. Then $w(g, z_0) \geqq \delta/2$ and g fails to be continuous at z_0 so g is in $B[a, b] - C[a, b]$. Hence, $B[a, b] - C[a, b]$ is open and $C[a, b]$ is closed in $B[a, b]$.

Reference

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NOTE ON $\int_0^\infty (\sin x/x)dx$

KENNETH S. WILLIAMS, Carleton University, Ottawa

In this note we give a simple proof of the well known result $\int_0^{+\infty} (\sin x/x)dx = \pi/2$. (All integrals used are either proper or improper Riemann integrals.) We do this by showing that it is a limiting form of a special case of the following result:

THEOREM 1. *If $f(x)$ is a real-valued function which is continuous on $[a, b]$, where $0 \leqq a < b$, then*

$$\int_0^{+\infty} \left\{ \int_a^b e^{-xy} x f(x) dx \right\} dy \text{ (exists) } = \int_a^b f(x) dx.$$

Proof. Since $f(x)$ is continuous on $[a, b]$, $f(x)$ is bounded on $[a, b]$, say $|f(x)| \leqq M$, for $x \in [a, b]$. Then for $y > 0$ we have

$$\begin{aligned} \left| \int_a^b (1 - e^{-xy}) f(x)dx - \int_a^b f(x)dx \right| &\leqq \int_a^b e^{-xy} \left| f(x) \right| dx \\ &\leqq M \frac{(e^{-ay} - e^{-by})}{y} \\ &\leqq \frac{M}{y}, \end{aligned}$$

so that

$$\lim_{y \to +\infty} \int_a^b (1 - e^{-xy}) f(x)dx = \int_a^b f(x)dx.$$

Now

$$1 - e^{-xy} = \int_0^y x e^{-ux} du,$$

for all x and y so that

$$\begin{aligned} \int_a^b (1 - e^{-xy}) f(x)dx &= \int_a^b \left\{ \int_0^y x e^{-ux} du \right\} f(x)dx \\ &= \int_0^y \left\{ \int_a^b e^{-ux} x f(x)dx \right\} du, \end{aligned}$$

since $e^{-ux}xf(x)$ is continuous on $[a, b] \times [0, y]$. Hence

$$\lim_{y \to +\infty} \int_0^y \left\{ \int_a^b e^{-ux} x f(x)dx \right\} du = \int_a^b f(x)dx,$$

which proves the result.

THEOREM 2. $\int_0^{+\infty} (\sin x/x)dx = \pi/2$.

Proof. In Theorem 1 we choose $a = 0$,

$$f(x) = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

so that $f(x)$ is continuous on $[0, b]$, for any $b > 0$. We obtain

$$\int_0^b \frac{\sin x}{x} dx = \int_0^\infty \left\{ \int_0^b e^{-ux} \sin x dx \right\} du$$

$$= \int_0^\infty \frac{1 - e^{-bu}(u \sin b + \cos b)}{1 + u^2} du$$

$$= \frac{\pi}{2} - \int_0^\infty \frac{e^{-bu}(u \sin b + \cos b)}{1 + u^2} du.$$

Now

$$\left| \int_0^\infty \frac{e^{-bu}(u \sin b + \cos b)}{1 + u^2} du \right| \leqq \int_0^\infty \frac{e^{-bu}}{\sqrt{1 + u^2}} du \leqq \int_0^\infty e^{-bu} du = \frac{1}{b},$$

so that letting $b \to +\infty$ we obtain the result.

A DEMOCRATIC PROOF OF A COMBINATORIAL IDENTITY

HARLEY FLANDERS, Tel Aviv University

In a note titled *A new proof of a combinatorial identity*, David C. Shipman gives a matrix theory proof of

$$\sum_{k=j}^{i} \binom{k}{j} \binom{i}{k} (-1)^{j+k} = 0 \quad \text{for } i > j.$$

See this MAGAZINE, 43 (1970) 162–163.

Suppose from an assembly of i individuals one appoints a committee of k and a subcommittee of j. This can be done in

$$\binom{i}{k}\binom{k}{j}$$

ways obviously. However, one could slyly appoint the subcommittee first and then select the other $k-j$ committeemen from the remaining $i-j$ assemblymen. Then the count is

$$\binom{i}{j}\binom{i-j}{k-j}.$$

Thus

$$\sum_{k=j}^{i} \binom{k}{j}\binom{i}{k} x^k = \binom{i}{j} \sum_{k=j}^{i} \binom{i-j}{k-j} x^k$$

$$= \binom{i}{j} x^j \sum_{r=0}^{i-j} \binom{i-j}{r} x^r = \binom{i}{j} x^j (1+x)^{i-j}.$$

Set $x = -1$.

ECONOMIC TRAVERSAL OF LABYRINTHS (CORRECTION)

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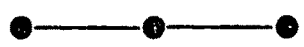
Bar-Ilan University, Ramat-Gan, Israel

Professor S. Even presented me with a counterexample to the algorithm given in [1]. The simplest counterexample is



0 1 2

The error in the proof is in (ii), where $0 \leqq i \leqq k$ should have been $0 \leqq i < k$. My thanks are due to Professor Even.

The algorithm can easily be restored by slightly modifying step (4) as follows:

(4) Suppose that the counter becomes zero at vertex v_k. If there is an untraversed edge incident to v_k, follow it. Otherwise, leave all vertices via their entrance edges.

(Note that traversing a vertex of valence $\leqq 2$ steps the counter up by (1) and down by (2), the net change being zero.)

In the proof of the theorem, (i) remains unchanged, (ii) is modified as follows:

(ii) Suppose that the counter becomes zero at vertex v_l. If there is an untraversed edge incident to v_l, the continuation of the journey will necessarily step the counter up. (The counter will remain positive at the next vertex, v_{l+1}, if $\rho(v_{l+1}) > 2$.) The argument of (i) now shows that the counter will again become zero.

Since the graph is finite, we will eventually reach a vertex v_k where the counter becomes zero such that all edges incident to v_k will have been traversed. By renaming vertices if necessary, we may suppose that the vertices $v_0, v_1, \cdots, v_k$ were traversed. Then *all* edges incident to $v_i (0 \leqq i \leqq k)$ must have been traversed. At this stage, all vertices of the graph have been traversed at least once. For suppose that W is a vertex not traversed even once. There is a simple path $v_0 = W_0, W_1, \cdots, W_{m-1}, W_m = W$ connecting v_0 with W. Let W_i be the first vertex in the sequence which was never traversed. Then W_{i-1} was traversed, and, in particular, the edge (W_{i-1}, W_i) was traversed; a contradiction.

The backtrack journey begins at v_k, during which the counter is not used. Thus the counter became zero at v_k for the last time.

The second sentence in (iii) should read: This is clear for the part of the excursion performed before the counter became zero for the last time, which is a subtour of Tarry.

The three expressions "counter became zero", "counter containing zero", "counter contains zero" which appear in the remainder of the proof and two similar expressions in remark (3), should all be replaced by "counter contains zero for the last time."

Reference

1. A. S. Fraenkel, Economic traversal of labyrinths, this MAGAZINE, 43 (1970) 125–130.

A REMARKABLE GROUP OF ANTIMAGIC SQUARES

CHARLES W. TRIGG, San Diego, California

In a 3-by-3 cellular array place 1 in the central cell, the sequence 3, 5, 7, 9 cyclically in the corner cells, and the sequence 2, 4, 6, 8 cyclically in the midside cells. Remarkably, whether the sequences run clockwise or counterclockwise, each of the eight essentially distinct squares thus obtained is *antimagic*. That is, in each square the sums of the eight digit-triads in the rows, columns, and unbroken diagonals are distinct. The squares are:

345	345	325	325	385	365	365	385
216	612	814	418	612	814	418	216
987	987	967	967	947	927	927	947

If 9 is placed in the central cell, and the odd and even sequences of the remaining digits are distributed cyclically as before, another set of eight antimagic squares is produced. This is the complement of the first set, obtainable by subtracting each digit element from 10.

When the sums of the four broken diagonals are also considered, exactly two duplicated sums appear among the twelve sums associated with each square. Consequently, each of the group of sixteen antimagic squares also is a *near heterosquare*, in the sense used by Charles F. Pinzka in *Heterosquares*, this MAGAZINE, 38 (September, 1965) 250–252.

TWO PROBLEMS ON MAGIC SQUARES

ELIZABETH H. AGNEW, Vassar College

Dr. Oystein Ore, some time before his death, had suggested the following two problems to the author who is submitting the answers thereto in his memory.

The first problem was submitted by Dr. Ore to the Pi Mu Epsilon Journal, fall, 1967, page 295.

PROBLEM NUMBER 192. Albrecht Durer's famous etching 'Melancholia' includes the magic square:

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

The boxed in numbers 15—14 indicate the year in which the picture was drawn. How many other 4×4 magic squares are there which he could have used in the same way?

This magic square is called a "regular" or "associated" magic square and has the following properties:

1. The sum of each row is 34.
2. The sum of each column is 34.
3. The sum of each of the main diagonals is 34.
4. The sum of any two numbers symmetric with respect to the middle of the square is $1/2 \times 34 = 17$.

With these properties in mind, we have this much to start with:

$$(a_{ij}) = \begin{bmatrix} \_ & 3 & 2 & \_ \\ \_ & \_ & \_ & \_ \\ \_ & \_ & \_ & \_ \\ x & 15 & 14 & y \end{bmatrix}$$

If we let $a_{41}=x$ and $a_{44}=y$, then $x+15+14+y=34$. Therefore:

$$x = 4 \quad \text{and} \quad y = 1, \quad \text{or} \quad x = 1 \quad \text{and} \quad y = 4, \quad \text{or}$$
$$x = 3 \quad \text{and} \quad y = 2, \quad \text{or} \quad x = 2 \quad \text{and} \quad y = 3.$$

But this last pair of possibilities must be eliminated because 2 and 3 must appear in the first line of (a_{ij}).

Thus, let us take the first possibility, $x=4$ and $y=1$. The entire first line is uniquely determined via the 4th property:

$$\begin{bmatrix} 16 & 3 & 2 & 13 \\ \_ & \_ & \_ & \_ \\ \_ & \_ & \_ & \_ \\ 4 & 15 & 14 & 1 \end{bmatrix}$$

How many different ways can the middle two lines be filled in using the numbers 5 through 12? A computer program was written to determine this. There are only 2 ways which will retain all the properties of the original square:

$$A = \begin{bmatrix} 16 & 3 & 2 & 13 \\ 5 & 10 & 11 & 8 \\ 9 & 6 & 7 & 12 \\ 4 & 15 & 14 & 1 \end{bmatrix} \qquad A_1 = \begin{bmatrix} 16 & 3 & 2 & 13 \\ 9 & 6 & 7 & 12 \\ 5 & 10 & 11 & 8 \\ 4 & 15 & 14 & 1 \end{bmatrix}$$

(This is the example given) (The middle two rows of A are exchanged)

By virtue of the computer program which tested every possibility, these are the only two arrangements that work.

The only remaining possibility is $x=1$ and $y=4$. The computer program yields the following two magic squares:

$$A_2 = \begin{bmatrix} \underline{13} & \underline{3} & \underline{2} & \underline{16} \\ \underline{8} & \underline{10} & \underline{11} & \underline{5} \\ \underline{12} & \underline{6} & \underline{7} & \underline{9} \\ \underline{1} & \underline{15} & \underline{14} & \underline{4} \end{bmatrix} \qquad A_3 = \begin{bmatrix} \underline{13} & \underline{3} & \underline{2} & \underline{16} \\ \underline{12} & \underline{6} & \underline{7} & \underline{9} \\ \underline{8} & \underline{10} & \underline{11} & \underline{5} \\ \underline{1} & \underline{15} & \underline{14} & \underline{4} \end{bmatrix}$$

(The outer columns of A are exchanged) (The middle two rows of A_2 are exchanged)

Thus, there are three other possible magic squares, A_1, A_2, and A_3, that Durer could have used in the same way he used A.

The second problem concerns the 16×16 Benjamin Franklin magic square.

200	217	232	249	8	25	40	57	72	89	104	121	136	153	168	185
58	39	26	7	250	231	218	199	186	167	154	135	122	103	90	71
198	219	230	251	6	27	38	59	70	91	102	123	134	155	166	187
60	37	28	5	252	229	220	197	188	165	156	133	124	101	92	69
201	216	233	248	9	24	41	56	73	88	105	120	137	152	169	184
55	42	23	10	247	234	215	202	183	170	151	138	119	106	87	74
203	214	235	246	11	22	43	54	75	86	107	118	139	150	171	182
53	44	21	12	245	236	213	204	181	172	149	140	117	108	85	76
205	212	237	244	13	20	45	52	77	84	109	116	141	148	173	180
51	46	19	14	243	238	211	206	179	174	147	142	115	110	83	78
207	210	239	242	15	18	47	50	79	82	111	114	143	146	175	178
49	48	17	16	241	240	209	208	177	176	145	144	113	112	81	80
196	221	228	253	4	29	36	61	68	93	100	125	132	157	164	189
62	35	30	3	254	227	222	195	190	163	158	131	126	99	94	67
194	223	226	255	2	31	34	63	66	95	98	127	130	159	162	191
64	33	32	1	256	225	224	193	192	161	160	129	128	97	96	65

Franklin 16×16 Square

The properties of this type of magic square are graphically illustrated in W. S. Andrews' *Magic Squares and Cubes*, Dover Publications, Inc., New York, 1960, pp. 104–105.

Dr. Ore suspected that there was a mistake somewhere in Franklin's square. He asked the author to write a program for the computer to see if this was so. The results are that Franklin made no mistake.

ANSWERS

A492. Yes, because $g(z)=f(z)-\overline{f(\bar{z})}$ is regular in the same region and $g(z)=0$ for $z=x>0$. Hence $g(z)=0$ for all z, so $f(x)=\overline{f(x)}$ for any real x.

A493. Let n be the number of triangular regions, and e, the number of edges interior to the given triangle. Of the $3n$ edges that the n triangles give rise to, 3 are the sides of the given triangle and the rest are interior edges, each of which is counted twice. Hence $e=\frac{1}{2}(3n-3)$. Since e is an integer, n must be odd.

A494. If we choose a rectangular coordinate system whose axes are along the three given chords, then the center is at the point $(b-a,\ d-c,\ f-e)$ where we are assuming without loss of generality that $b \geqq a$, $d \geqq c$ and $f \geqq e$. Then

$$R^2 = (b - a - 2b)^2 + (d - c - 0)^2 + (b - e - 0)^2$$
$$= a^2 + b^2 + c^2 + d^2 + e^2 + f^2 - 2ef$$
$$(\text{since } ab = cd = ef).$$

It is to be noted that the result is easily extended for the case of an n-dimensional sphere. For the special case of the circle ($n=2$), the cross terms disappear.

A495. $1/\cot^2\theta=\tan^2\theta$, $1+\tan^2\theta=\sec^2\theta$, and $1/\sec^2\theta=\cos^2\theta$. Since the cotangents squared are in harmonic progression, the tangents squared and the secants squared are in arithmetic progression, and the cosines squared are in harmonic progression. The same statement may be made about the pairs $\sin^2\theta$ and $\tan^2\theta$, and $\sec^2\theta$ and $\csc^2\theta$, and conversely.

A496. The only interest of this problem is that most texts imply that it can't be done.

An example is:

$$y_1 = \begin{cases} x^2 & \text{for all } x \leqq 0 \\ 0 & \text{for all } x > 0 \end{cases}$$

$$y_2 = \begin{cases} 0 & \text{for all } x \leqq 0 \\ x^2 & \text{for all } x > 0 \end{cases}$$

The two functions y_1 and y_2 are continuous, everywhere differentiable, and defined over the same range. One is not a linear combination of the other. Yet:

$$W = \begin{vmatrix} x^2 & 0 \\ 2x & 0 \end{vmatrix} \quad \text{for all } x \leqq 0$$

and

$$W = \begin{vmatrix} 0 & x^2 \\ 0 & 2x \end{vmatrix} \quad \text{for all } x > 0$$

A497. Consider the equations as a set of homogeneous equations:

$$a_iX + b_iY + c_iZ = 0$$

where

$$x = X/Z, \qquad y = Y/Z.$$

If a system of homogeneous linear equations in n unknowns has a set of solutions other than the trivial one in which each unknown is zero, then the determinant of the coefficients of the unknowns is zero. Hence the condition for concurrency is

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$$

A498. Divide the triangle perimeter into eleven equal parts and make vertical cuts emanating from the center of the inscribed circle to these points of division. This problem for a square appears in H. S. M. Coxeter, *Introduction to Geometry*, and the method is valid for any polygonal layer cake having an incircle.

A499. Recall that a topological space X is locally connected if the components of open sets are open: Thus the components of U are open sets in E'. But components, being connected, are simply intervals in E', whence U is the union of disjoint open intervals. Finally, since each open interval contains a distinct rational number, the union must be countable.

A500.
$$\begin{aligned} 4x^3 + 6x^2 + 4x + 1 &= (x+1)^4 - x^4 \\ &= [(x+1)^2 - x^2][(x+1)^2 + x^2] \\ &= (2x+1)[(x+1)^2 + x^2]. \end{aligned}$$

A501. Let $c = \prod_{i=1}^{7} \cos(r_i\pi/15)$ and $s = \prod_{i=1}^{7} \sin(r_i\pi/15)$. Then

$$\begin{aligned} 2^7 cs &= \prod_{i=1}^{7} [2\cos(r_i\pi/15)\sin(r_i\pi/15)] \\ &= \prod_{i=1}^{7} \sin(2r_i\pi/15). \end{aligned}$$

But

$$\begin{aligned} \sin(8\pi/15) &= \sin(\pi - 7\pi/15) \\ &= \sin(7\pi/15) \end{aligned}$$

Similarly

$$\sin(10\pi/15) = \sin(5\pi/15)$$
$$\sin(12\pi/15) = \sin(3\pi/15)$$

and

$$\sin(14\pi/15) = \sin(\pi/15).$$

Therefore $2^7 cs = s$. Consequently $c = 1/2^7 = (1/2)^7$.

A502. Define $g(x)=(x^a-1)f(x)$. On $[0, 1]$, y satisfies the hypothesis of Rolle's theorem. Therefore there exists $c\in(0, 1)$ such that

$$0 = g'(c) = ac^{a-1}f(c) + (c^a - 1)f'(c)$$

which yields

$$f'(c) = \frac{ac^{a-1}f(c)}{1 - c^2} \cdot$$

A503. Place the dog at any point between the positions of the boy and girl after one hour and facing either direction. Let all three reverse their motion until they come together at the starting point at the starting time. Thus the answer is that the dog may be anywhere between the two and facing either direction.

A504. Cardinality$=c$ (See Casper Goffman, *Real Functions*). Note that the Cantor set has irrational numbers such as $2/3+0/9+ \cdots +a_n/3^n+ \cdots$ where a_n is 0 or 2, randomly chosen.

A505. Apply Leibnitz' rule to

$$[(x^2 - ax - bx - ab)y]''$$

to produce our differential equation and a solution if $y=1/(x-a)(x-b)$. Hence the solution is $y=A/(x-a)+B/(x-b)$ with validity in the interval $|x|<$the lesser of $|a|$ and $|b|$.

A506. Let $\overline{P}$ and $\overline{Q}$ denote the perimeters of P and Q respectively. Extend both ways any side of P which is not a side of Q and denote its first two intersections with A by A and B. Then from Q we create a new polygon Q, still containing P, in which a part of Q has been replaced by a definitely shorter segment AB. Next we take another side of P and create another Q_2 from Q_1, and so on. Thus we obtain a finite sequence of polygons such that $Q>Q_1>Q_2 \cdots >Q_n=P$. Therefore $Q>P$.

The extension to higher dimensions is obvious.

A507. If $(N^2-71)/(7N+55)$ is an integer M then $N^2-7MN-(55M+71)=0$. Solving as a quadratic in N, the radicand must be a perfect square and since

$$\begin{aligned}(7M + 15)^2 &= 49M^2 + 210M + 225\\ &< 49M^2 + 220M + 284\\ &< 49M^2 + 238M + 289\\ &= (7M + 17)^2\end{aligned}$$

the radicand must be $(7M+16)^2$, giving $M=7$ and $N=57$ or -8.

A508. Since the center of the required sphere of radius r is on a diagonal of the box $16\sqrt{3}-15-r$ inches from the nearest corner, we need only to put this value equal to $r\sqrt{3}$ and solve for r, getting $D=(16\sqrt{3}-15)(\sqrt{3}-1)=63-31\sqrt{3}$ $=9.308$ inches, the diameter of the required sphere.

A509. $$\sum_{k=0}^{n} \frac{k^2+3k+1}{(k+2)!} = \sum_{k=0}^{n} (1/k! - 1/(k+2)!)$$

$$= \left(\frac{1}{0!} - \frac{1}{2!}\right) + \left(\frac{1}{1!} - \frac{1}{3!}\right) + \cdots + \left(\frac{1}{n!} - \frac{1}{(n+2)!}\right)$$

$$= \frac{1}{0!} + \frac{1}{1!} - \frac{1}{(n+1)!} - \frac{1}{(n+2)!}$$

$$= 2 - \frac{n+3}{(n+2)!}.$$

It follows that the limit is 2.

A510. Let 1, a, b, $\cdots n$ be the divisors of n in increasing order and suppose $1/1+1/a+1/b+\cdots+1/n=2$. Multiplying through by n we have $n/a+n/b+\cdots+1=n$, where the left number contains the proper divisors of n in decreasing order. By definition then, n is perfect, and the next two perfect numbers are 28 and 496.

A511. Clearly $E^{(n)}(x)=E(x)$ for all x. But the Taylor series expansion of $E(x+y)$ is

$$\sum_{n=0}^{\infty} \frac{E^n(x)y^n}{n!} = E(x)\sum_{n=0}^{\infty} y^n/n! = E(x)E(y).$$

A512. By taking the natural logarithm of the equation we obtain the equivalent equation $f(x)=x\ln x + \ln(x+1)=0$. The graph is monotonically increasing throughout its range $(x>0)$, hence there can exist no more than one real root. Routine interpolation in tables of natural logarithms leads to $x \doteq 0.43605$.

(Quickies on pages 54–56.)

A NOTE ON THE SEQUENCE OF FIBONACCI NUMBERS

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In the work of Wall the sequence of Fibonacci numbers reduced modulo m was considered and shown to be periodic. Some interesting bounds for the period were obtained, but except in specific examples, the period was never exactly identified. By elementary means we have been able to find the period when the sequence of Fibonacci numbers is reduced modulo a term of itself.

In fact, we prove our results for a generalized Fibonacci sequence. We define the sequence $\{f_n\}$ of *t-Fibonacci* numbers for any positive integer t as follows:

$$f_0 = 0, \qquad f_1 = 1, \qquad f_2 = t,$$

$$f_{n+1} = tf_n + f_{n-1} \qquad (n \geqq 2).$$

Thus, when $t=1$ we obtain the well-known sequence of Fibonacci numbers.

Clearly, if t_1 and t_2 are positive integers with $t_1>t_2$ then the sequence of t_1-Fibonacci numbers is term-by-term greater than the sequence of t_2-Fibonacci numbers, and obviously these are increasing sequences.

The sequence of t-Fibonacci numbers arises in calculating the powers of the unimodular matrix

$$U = \begin{pmatrix} t & 1 \\ 1 & 0 \end{pmatrix},$$

an easy induction argument showing that if m is a positive integer then

$$U^m = \begin{pmatrix} f_{m+1} & f_m \\ f_m & f_{m-1} \end{pmatrix}.$$

We use U to prove some simple properties of the sequence $\{f_n\}$.

LEMMA 1. (i) *If m is an integer greater than 3 then $f_m-f_{m-1}>1$. If m is any positive integer then*

(ii) $f_m^2 - f_{m-1}f_{m+1} = (-1)^{m-1}$.

(iii) $(f_m, f_{m+1}) = 1$.

(iv) $f_{m+n} = f_{m+1}f_n + f_mf_{n-1} \qquad (n \geqq 1)$.

(v) *f_m divides f_{rm} for* $r = 1, 2, 3, \cdots$.

Proof. (i) This is clear if $t=1$, so suppose that $t>1$. Then

$$f_m - f_{m-1} = tf_{m-1} + f_{m-2} - f_{m-1} > f_{m-2} \geqq f_2 = t > 1.$$

(ii) Since $\det U = -1$ we immediately see the truth of this result.

(iii) This follows directly from (ii).

(iv) The equality $U^{m+n-1} = U^m U^{n-1}$ proves this result.

(v) When $r=1$ the result is trivially true. Suppose that $r>1$ and that f_m divides $f_{(r-1)m}$. Using (iv) we have

$$f_{rm} = f_{m+1}f_{(r-1)m} + f_mf_{(r-1)m-1},$$

and so the induction hypothesis shows that f_m divides f_{rm}.

In what follows, the symbol $\equiv$ indicates modulo congruence f_m where m is an integer greater than 3.

LEMMA 2. *If s is an integer with $1 \leqq s \leqq m-1$ then*

(i) $$f_{m+s} \equiv \begin{cases} f_{m-s} & \textit{if } s \textit{ is odd} \\ f_m - f_{m-s} & \textit{if } s \textit{ is even.} \end{cases}$$

If m is an odd integer then

(ii) $f_{2m+s} \equiv f_m - f_s$.

(iii) $$f_{3m+s} \equiv \begin{cases} f_m - f_{m-s} & \textit{if } s \textit{ is odd} \\ f_{m-s} & \textit{if } s \textit{ is even.} \end{cases}$$

Proof. We prove all these results by induction on s.

(i) Since

$$f_{m+1} = tf_m + f_{m-1} \equiv f_{m-1},$$

the result is true for $s=1$. It is also true for $s=2$ because

$$\begin{aligned} f_{m+2} &= tf_{m+1} + f_m \\ &\equiv tf_{m-1} \qquad \text{by the case } s = 1 \\ &= f_m - f_{m-2}. \end{aligned}$$

Suppose the result is true for s and $s-1$ where $s \geqq 2$. If $s+1$ is odd then

$$\begin{aligned} f_{m+s+1} &= tf_{m+s} + f_{m+s-1} \\ &\equiv tf_m - tf_{m-s} + f_{m-s+1} \quad \text{by hypothesis} \\ &= tf_m + f_{m-s-1} \\ &\equiv f_{m-s-1}. \end{aligned}$$

If $s+1$ is even then

$$\begin{aligned} f_{m+s+1} &= tf_{m+s} + f_{m+s-1} \\ &\equiv tf_{m-s} + f_m - f_{m-s+1} \quad \text{by hypothesis} \\ &= f_m - f_{m-s-1}. \end{aligned}$$

(ii) We have

$$\begin{aligned} f_{2m+1} &= tf_{2m} + f_{2m-1} \\ &\equiv f_{2m-1} \quad \text{by Lemma 1 (v)} \\ &\equiv f_m - f_1 \quad \text{by(i)}. \end{aligned}$$

Thus the result is true if $s=1$. Also it is true if $s=2$ because

$$\begin{aligned} f_{2m+2} &= tf_{2m+1} + f_{2m} \\ &\equiv tf_m - tf_1 + f_{2m} \quad \text{by the case } s = 1 \\ &\equiv f_m - f_2 \quad \text{by Lemma 1 (v) and because } f_2 = t. \end{aligned}$$

Suppose the result is true for s. Then

$$\begin{aligned} f_{2m+s+1} &= tf_{2m+s} + f_{2m+s-1} \\ &\equiv tf_m - tf_s + f_m - f_{s-1} \quad \text{by hypothesis} \\ &\equiv f_m - f_{s+1} \end{aligned}$$

(iii) Using (ii) and Lemma 1 (v) we easily obtain the truth of this result when $s=1$ and $s=2$. Suppose the result is true for $s \geqq 2$. If $s+1$ is odd then

$$\begin{aligned} f_{3m+s+1} &= tf_{3m+s} + f_{3m+s-1} \\ &\equiv tf_{m-s} + f_m - f_{m-s+1} \quad \text{by hypothesis} \\ &= f_m - f_{m-s-1}. \end{aligned}$$

If $s+1$ is even then

$$\begin{aligned} f_{3m+s+1} &= tf_{3m+s} + f_{3m+s-1} \\ &\equiv tf_m - tf_{m-s} + f_{m-s+1} \quad \text{by hypothesis} \\ &\equiv f_{m-s-1}. \end{aligned}$$

We may now prove:

THEOREM. *If m is an integer greater than 3 then the sequence of t-Fibonacci numbers reduced modulo f_m to least nonnegative residues is periodic. If $\phi(f_m)$ denotes the period of the series then $\phi(f_m)=2m$ if m is even and $\phi(f_m)=4m$ if m is odd.*

Proof. Suppose that m is even. Then by Lemma 2 (i) the sequence of t-Fibonacci numbers reduced modulo f_m consists of repetitions of the numbers

$$f_0, f_1, f_2, \cdots, f_{m-1}, 0, f_{m-1}, f_m - f_{m-2}, f_{m-3}, f_m - f_{m-4}, \cdots, f_3, f_m - f_2, f_1.$$

Since $f_{m-1} \neq f_1$ for $m>3$ we find that $\phi(f_m)=2m$.

If m is odd then by Lemma 2 we obtain the sequence

$$\begin{aligned} &f_0, f_1, f_2, \cdots, f_{m-1}, 0, f_{m-1}, f_m - f_{m-2}, f_{m-3}, \cdots, f_m - f_3, f_2, \\ &f_m - f_1, 0, f_m - f_1, f_m - f_2, \cdots, f_m - f_{m-1}, 0, f_m - f_{m-1}, f_{m-2}, \\ &f_m - f_{m-3}, \cdots, f_m - f_2, f_1, \end{aligned}$$

and the sequence of t-Fibonacci numbers reduced modulo f_m consists of repetitions of this. But for $m>3$ we have $f_{m-1}\neq f_1$, $f_m - f_1 \neq f_1$ and $f_m - f_{m-1} \neq f_1$, the latter by Lemma 1 (i). Thus we have that $\phi(f_m)=4m$, and this proves the theorem.

It is interesting to note that in each case $\phi(f_m)$ is independent of t.

The cases not covered by the theorem are easily checked. In fact, we have $\phi(f_2)=2$ and $\phi(f_3)=12$ if $t\neq 1$, while if $t=1$ we have $\phi(f_3)=3$.

Now suppose that $t=1$, so that we are dealing with the sequence of Fibonacci numbers. If we apply Wall's Theorem 6 we find that if f_m is a prime of the form $10x+1$, so that, of necessity $m>3$, then $\phi(f_m) \mid (f_m - 1)$.

Now f_m is a prime only when $m=4$ or m is an odd prime. But $f_4=3$ is not of the required form, so that m is an odd prime. Thus, by the theorem, we have that $4m \mid 10x$. This means that x is an even integer, say $x=2y$, and then $m \mid 5y$. Thus $m=5$ or $m \mid y$. But $f_5=5$ is not of the required form, and so $m \mid y$. We have thus proved the corollary:

For prime Fibonacci numbers f_m of the form $10x+1$ we must have $f_m=20km+1$ for some integer k.

Other results of this nature hold if f_m is a prime of the form $10x-1$ or $10x\pm 3$, although they are less easily stated.

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A NOTE ON THE LOGARITHMIC AND BINOMIAL EXPANSIONS

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Consider only real x and μ, and let

$$u_n(x) = \binom{\mu-1}{n-1}\frac{x^n}{n}, \qquad l(x) = \begin{cases} \log(1+x), x > -1 \ (\mu = 0), \\ \dfrac{(1+x)^\mu - 1}{\mu} \begin{cases} x > -1 (\mu \neq 0), \\ x = -1 (\mu > 0). \end{cases} \end{cases}$$

Combining the logarithmic and binomial expansions, we have

$$(1) \qquad l(x) = \sum_{n=1}^{\infty} u_n(x) \begin{cases} |x| < 1 & (\mu \text{ arbitrary}), \\ x = 1 & (\mu > -1), \\ x = -1 & (\mu > 0). \end{cases}$$

It may be interesting to note that the simple identity

$$(*) \qquad \sum_{k=0}^{n}\binom{\mu}{k}x^k - (1+x)\sum_{k=1}^{n}\binom{\mu-1}{k-1}x^{k-1} = \binom{\mu-1}{n}x^n,$$

which merely asserts that

$$\binom{\mu}{k} = \binom{\mu-1}{k} + \binom{\mu-1}{k-1},$$

is capable of yielding the entire proof of (1). Special cases of it are already familiar in this connection: while the case $\mu = n$ leads to the binomial identity, the cases $|x| < 1$ $(n \to \infty)$ and $x = -1$, together with the limit

$$(2) \qquad \lim_{n\to\infty} u_n(x) = \lim_{n\to\infty}\binom{\mu}{n}x^n = 0 \begin{cases} |x| < 1 & (\mu \text{ arbitrary}), \\ |x| = 1 & (\mu > -1), \end{cases}$$

establish the first and the last cases of (1) respectively. With

$$R_n(x) = l(x) - \sum_{k=1}^{n-1} u_k(x), \qquad x > -1,$$

we now exploit (*) to obtain the estimate

$$(**) \qquad R_n(x) = (1+x)^\mu(1+\theta_n x)^{-\mu-1}u_n(x), \qquad 0 < \theta_n < 1.$$

The first two cases of (1) follow at once from this by (2). In their usual treatment by Taylor's theorem, the first case is derived from Cauchy's estimate

$$R_n(x) = n(1-\alpha_n)^{n-1}(1+\alpha_n x)^{\mu-n}u_n(x), \qquad 0 < \alpha_n < 1,$$

and the second from Lagrange's estimate

$$R_n(x) = (1+\beta_n x)^{\mu-n}u_n(x), \qquad 0 < \beta_n < 1,$$

neither of these estimates being strong enough to cover both the cases.

For the desired proof of (**), we have just to observe that (*) with n

changed to $n-1$ may be written

$$(1+x)R_n'(x) - \mu R_n(x) = u_n'(x), \qquad x > -1,$$

which, on setting

$$r_n(x) = (1+x)^{-\mu} R_n(x),$$

can be replaced by

$$r_n'(x) = (1+x)^{-\mu-1} u_n'(x), \qquad x > -1. \tag{3}$$

An integration yields

$$r_n(x) = u_n(x) \int_0^1 n t^{n-1} (1+xt)^{-\mu-1} dt \tag{4}$$

and the result is immediate. For $\mu = 0$, when (3) is obvious, this is well known. One readily verifies that (4) is the transformation of Taylor's formula

$$R_n(x) = u_n(x) \int_0^1 n(1-\tau)^{n-1}(1+x\tau)^{\mu-n} d\tau$$

by the change of variable $\tau = (1-t)/(1+xt)$.

In deducing (* *) from (3), we can easily avoid integrals with the help of Rolle's theorem. Fix $x > -1$ and consider the function

$$f_n(t) = r_n(xt) - t^n r_n(x), \qquad 0 \leqq t \leqq 1.$$

Since $f_n(0) = 0 = f_n(1)$ and, by (3),

$$f_n'(t) = n t^{n-1} \{ (1+xt)^{-\mu-1} u_n(x) - r_n(x) \}, \qquad 0 \leqq t \leqq 1,$$

Rolle's theorem applies to f_n on [0, 1], giving the result.

SUMMATION OF SERIES BY THE RESIDUE THEOREM

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The usual application of the calculus of residues to the summation of series does not seem very suitable for a first course in complex variables. The student is usually "turned off" by the artificial use of functions such as $\pi \cot \pi z$ and by the estimations involved. See, for example, [1] or [2].

We have found that particular cases of the theorem given below provide simple and useful examples of the residue theorem applied to summation, although these examples are somewhat different in spirit from the usual ones. While results of this kind are not new, they seem to have been neglected by most modern textbook writers. This theorem and similar theorems can be derived from expansion theorems due to Lagrange and Bürmann [3], [4], [5]. In this note, the result is obtained directly from the residue theorem and Rouché's theorem.

We are indebted to Professor Donald J. Newman of Yeshiva University for calling such applications to our attention.

THEOREM. *If* $|z| < (k-1)^{k-1}/k^k$, *then*

$$\sum_{n=0}^{\infty} \binom{kn}{n} z^n = \frac{1+W}{1-(k-1)W},$$

where W *is the unique root of the equation* $w - z(1+w)^k = 0$ *inside the circle* $|w| = 1/(k-1)$.

Proof. The crucial observation is that $\binom{kn}{n}$ is the coefficient of w^n in the expansion of $(1+w)^{kn}$. Dividing this last expression by w^{n+1} gives us a function having a multiple pole at $w=0$ with residue $\binom{kn}{n}$. Of course, the Cauchy integral formula could have been used here.

Now, if C is a simple closed contour enclosing the origin, the residue theorem yields

$$(*) \qquad \sum_{n=0}^{\infty} \binom{kn}{n} z^n = \sum_{n=0}^{\infty} \left\{ \frac{1}{2\pi i} \int_C \frac{(1+w)^{kn}}{w^n} \frac{dw}{w} \right\} z^n$$
$$= \frac{1}{2\pi i} \int_C \sum_{n=0}^{\infty} \left(\frac{(1+w)^k z}{w} \right)^n \frac{dw}{w},$$

where the interchange of summation and integration is justified if the sum in the integrand converges uniformly on C.

Since $|z| < (k-1)^{k-1}/k^k$, it is easy to verify that the summand is less than 1 in absolute value (for $n > 1$) if C is the circle $|w| = 1/(k-1)$.

Summing this geometric progression on C, we find that (*) becomes

$$\frac{1}{2\pi i} \int_C \frac{dw}{w - z(1+w)^k}.$$

By Rouché's theorem, the equation $w - z(1+w)^k = 0$ has a unique root W inside C; and the theorem follows by using the residue theorem.

This theorem can be used effectively in the classroom by considering only the values $k = 1, 2, 3, 4$—that is, those values of k for which the equation $w - z(1+w)^k = 0$ is solvable by radicals.

For example, choosing $k=2$ and $z = 1/5$, we solve a quadratic equation for its only root within the unit circle and find that the series has the sum $\sqrt{5}$. The values $k=2$ and $z=1/10$ give a series with sum $\sqrt{15}/3$, while the values $k=3$ and $z=2/27$ give a series with sum equal to $(\sqrt{3}+1)/2$. Of course, negative and complex values of z can be used, provided that they lie within the series' circle of convergence.

For any values of k and z chosen, it is instructive to discuss the selection of the contour of integration and the interchange of summation and integration.

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A GENERATING PROPERTY OF PYTHAGOREAN TRIPLES

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In [1] the authors indicate a generating property of pairs of Pythagorean triples. Given the triples (a, b, c) and (x, y, z) then $((ax-by), (ay+bx), cz)$ is a Pythagorean triple. What we wish to establish in this note is a generating property of any Pythagorean triple. We do this by means of two propositions stated below. First, a preliminary.

Let (a, b, c) be a Pythagorean triple. One of a or b must have the same parity as c. We shall assume that b and c have the same parity, where necessary.

PROPOSITION 1. *Let (a, b, c) be a Pythagorean triple and $d=c-b$. For n a positive integer define $a_1=a$, $b_1=b$, $c_1=c$ and $a_n=a_{n-1}+2d$, $b_n=a_n+a_{n-1}+b_{n-1}$, $c_n=b_n+d$. Then (a_n, b_n, c_n) is a Pythagorean triple. Further, if (a, b, c) is a primitive triple then (a_n, b_n, c_n) is a primitive triple.*

Proof. If $n=1$ the proposition is trivially true. Suppose that the proposition is true for all positive k less than n. Now $a_n^2+b_n^2=(a_{n-1}+2d)^2+(2a_{n-1}+b_{n-1}+2d)^2$. On expanding, rearranging terms and using the fact that $a_{n-1}^2+b_{n-1}^2=c_{n-1}^2=(b_{n-1}+d)^2$ we have $a_n^2+b_n^2=c_n^2$. Now suppose that for some positive k, (a_k, b_k, c_k) is not a primitive triple while (a, b, c) is a primitive triple. Let p be any prime divisor of a_k, b_k, c_k. Since $d=c_k-b_k$, p divides d. Further, since $a_{k-1}=a_k+2d$, p divides a_{k-1}. But then p divides b_{k-1} since $b_{k-1}=b_k-(a_{k-1}+a_k)$. And finally, since $c_{k-1}=b_{k-1}+d$, p divides c_{k-1}. Surely then p divides each of a, b, c contradicting our hypothesis.

PROPOSITION 2. *Let (a, b, c) be a Pythagorean triple and $d=c-b$. For n a positive integer, define $a_1=a$, $b_1=b$, $c_1=c$ and $a_n=a_{n-1}+d$, $b_n=a_{n-1}+b_{n-1}+d/2$, $c_n=b_n+d$. Then (a_n, b_n, c_n) is a Pythagorean triple for all n. Further, if $d/2$ is even and (a, b, c) is primitive then (a_n, b_n, c_n) is a primitive triple for all n.*

Proof. If $n=1$ the proposition is trivially true. Hence suppose that the proposition is true for all positive k less than n. Now $a_n^2+b_n^2=(a_{n-1}+d)^2+(a_{n-1}+b_{n-1}+d/2)^2$. On expanding, rearranging terms and using the fact that $a_{n-1}^2+b_{n-1}^2=c_{n-1}^2=(b_{n-1}+d)^2$ we have $a_n^2+b_n^2=c_n^2$. Now suppose that for positive k, (a_k, b_k, c_k) is not a primitive triple while (a, b, c) is primitive and $d/2$ is even. Let p be any prime divisor of a_k, b_k, c_k. Now p divides d and p divides a_{k-1}. If p is odd, then p divides $d/2$. Hence p divides b_{k-1}. But then p divides c_{k-1}. Hence surely p divides a, b, c contradicting our hypothesis. If $p=2$ and $d/2$ is even then 2 divides b_{k-1} since $b_{k-1}=b_k-(a_{k-1}+d/2)$. And again 2 divides c_{k-1}. Hence we conclude that 2 divides a, b, c. Again a contradiction.

A scheme for generating Pythagorean triples should now be obvious.

Reference

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A SIMPLER PROOF OF HERON'S FORMULA

CLAUDE H. RAIFAIZEN, M. I. T.

A few years ago while attempting a proof of Heron's formula for my own amusement, I discovered one which, compared to Heron's complicated geometrical proof and to the trigonometric proofs for it, is quite simple.

In any triangle, for at least one of the vertices, the perpendicular from that vertex to the line containing the opposite side intersects that side. In what follows we will assume that in triangle ABC one such vertex is C. (See Figure 1.)

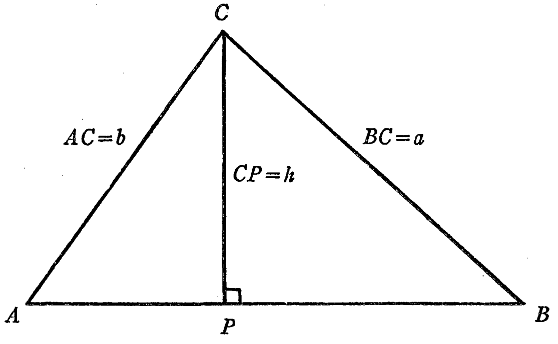

FIG. 1.

(1) Letting $CP=h$ we see that by the Pythagorean formula $AP=\sqrt{b^2-h^2}$, $PB=\sqrt{a^2-h^2}$, and hence

$$c = AB = AP + PB = \sqrt{b^2-h^2} + \sqrt{a^2-h^2}$$

(2) $\sqrt{a^2-h^2} = c - \sqrt{b^2-h^2}$; $\quad a^2-h^2 = c^2 - 2c\sqrt{b^2-h^2} + b^2 - h^2$;

$$a^2-b^2-c^2 = -2c\sqrt{b^2-h^2}; \quad (a^2-b^2-c^2)^2 = 4c^2(b^2-h^2) = 4b^2c^2 - 4c^2h^2.$$

(3) Observing that $ch=2A$, where A is the area, we have $(a^2-b^2-c^2)^2 = 4b^2c^2-16A^2$.

(4) Then

$$\begin{aligned} 16A^2 &= 4b^2c^2 - (a^2 - b^2 - c^2)^2 \\ &= (2bc - a^2 + b^2 + c^2)(2bc + a^2 - b^2 - c^2) \\ &= [(b^2 + 2bc + c^2) - a^2][a^2 - (b^2 - 2bc + c^2)] \\ &= [(b + c)^2 - a^2][a^2 - (b - c)^2] \\ &= (b + c + a)(b + c - a)(a - b + c)(a + b - c) \\ &= (a + b + c)(a + b + c - 2a)(a + b + c - 2b)(a + b + c - 2c). \end{aligned}$$

(5) Letting $a+b+c=2s$ we get $16A^2=(2s)\ (2s-2a)\ (2s-2b)\ (2s-2c)$ and

$$A^2 = s(s - a)(s - b)(s - c).$$

ON "ROTATING" ELLIPSES INSIDE TRIANGLES

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What we mean by roundness, or circularity, is fairly straightforward. However, when we attempt to devise criteria or tests for roundness, our first intuitive impressions can be quite erroneous. For example, one might consider the following experiment a suitable test to check the roundness of a curve (or the cross section of an object). The curve is placed in a V-groove, and a horizontal bar is lowered until it is in contact with the curve. The height of the horizontal bar, H, is noted. If the apex of the V-groove is taken as the origin for the height measurement, then H is just the height of the triangle formed by the V-groove and the horizontal bar. If we now rotate the curve in the V-groove, and observe that the value of H changes as the orientation of the curve is changed, then we would certainly conclude that the figure was not round. However, the converse is not true; there do exist certain noncircular curves for which the horizontal bar would always have the same height regardless of their orientation.

The ability of the curve to alter its orientation in the V-groove without altering the height of the horizontal bar is equivalent to the curve's being able to "rotate" inside a triangle. By "rotate" we mean that it can be smoothly turned while always remaining in contact with every side of the triangle. Curves which possess this ability to rotate inside various figures have been extensively studied. In a survey paper Michael Goldberg [1] lists thirty-five references to original work. Others who have discussed the problem in a general way include Martin Gardner [2] who cites one example of a curve which can rotate inside an equilateral triangle. This is formed by two arcs of circles with radii equal to the altitude of the triangle, resulting in a lens-shaped figure.

However, despite the vast literature on the subject, the case of the "simplest" smooth curve which can depart from exact circularity by arbitrarily small amounts, the ellipse, seems to have been overlooked. Not only is the ellipse of obvious physical importance as a likely approximation to the result of an attempt to produce a circular cross section, but it is also an interesting self-contained example of the problems encountered in roundness gauging [3]. In addition, it can be treated using only elementary mathematics.

The present writer first became interested in the problem by performing the following simple experiment. Take an accurate ellipse (commercially available) and, using straight edges, construct an equilateral triangle around it. Hold the edges of the triangle fixed and carefully turn the ellipse. It is quite remarkable to observe how the ellipse rotates, always remaining, as far as the eye and sense of touch can tell, in perfect contact with all sides of the triangle.

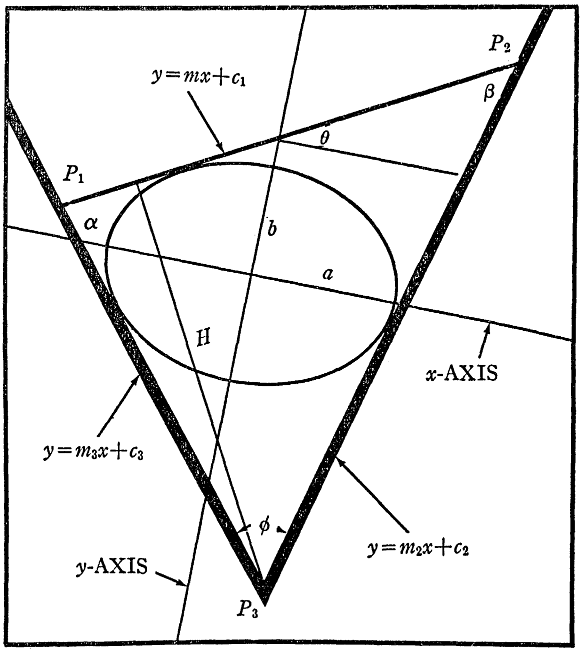


FIG. 1. Ellipse in V-groove.

We now investigate analytically what happens when an ellipse is rotated in a V-groove. Figure 1 shows a V-groove, apex P_3, angle ϕ, into which an ellipse, semimajor axis $=a$ and semiminor axis $=b$, has been placed in an arbitrary configuration. For the time being we will not constrain the "horizontal bar" P_1P_2 to be horizontal, but rather let $P_1P_2P_3$ be a general scaline triangle, and obtain H, the perpendicular distance from P_3 to P_1P_2 in terms of the angles α, β, ϕ of the triangle and the orientation of the ellipse. Applying the law of sines we obtain immediately

$$H = \frac{\sin \alpha \sin \beta}{\sin \phi} \times P_1P_2. \tag{1}$$

To obtain the length P_1P_2 we adopt a coordinate frame defined by the axes of the ellipse. Let P_1P_2 make an angle θ with the x-axis (and therefore with the major axis of the ellipse). Define

(2) $$m = \tan\theta.$$

It follows that

(3) $$P_1P_2 = (1 + m^2)^{1/2} | X_1 - X_2 |$$

where X_1 and X_2 are the abscissae of P_1 and P_2 respectively. From the equations shown in Figure 1 for the sides of the triangle it follows that

(4) $$X_1 - X_2 = \frac{c_3 - c_1}{m - m_3} - \frac{c_2 - c_1}{m - m_2}.$$

The slopes m_2 and m_3 are readily shown to be

(5) $$m_2 = \frac{ms + 1}{s - m}$$

(6) $$m_3 = \frac{mr - 1}{r + m}$$

where

(7) $$r = \cot\alpha$$

and

(8) $$s = \cot\beta.$$

The y-axis intercepts, c_i, are given by

(9) $$c_i = \pm (b^2 + m_i^2a^2)^{1/2} \qquad (i = 1, 2, 3; m_1 = m).$$

Combining equations 5, 6, 9, 4, 3 and 1 leads to

$$H = H(\alpha, \beta, \phi, a, b, \theta)$$

(10) $$= \frac{\sin\alpha \sin\beta}{\sin\phi\sqrt{1 + m^2}} \left[(r + s)\sqrt{b^2 + m^2a^2} + \left\{b^2(s - m)^2 + a^2(ms + 1)^2\right\}^{1/2} + \left\{b^2(r + m)^2 + a^2(mr - 1)^2\right\}^{1/2}\right].$$

To see what happens as we rotate the ellipse, let us define

(11) $$\Delta H(\theta) = H(\theta) - H(0).$$

If we consider the case of an isosceles triangle with $\alpha = \beta = 90° - \phi/2$, we have that

(12) $$\Delta H(90°) = a - b + (a^2 + b^2x^2)^{1/2} - (b^2 + a^2x^2)^{1/2}$$

where

(13) $$x = \cot(\phi/2).$$

For any ellipse there is a value of ϕ for which equation 12 is equal to zero. This

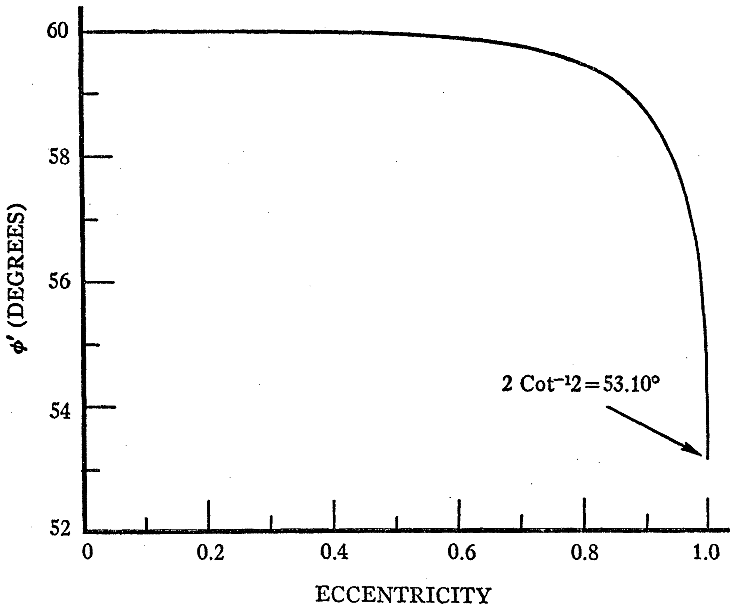


FIG. 2. Angle (ϕ') which $H(0)=H(90°)$.

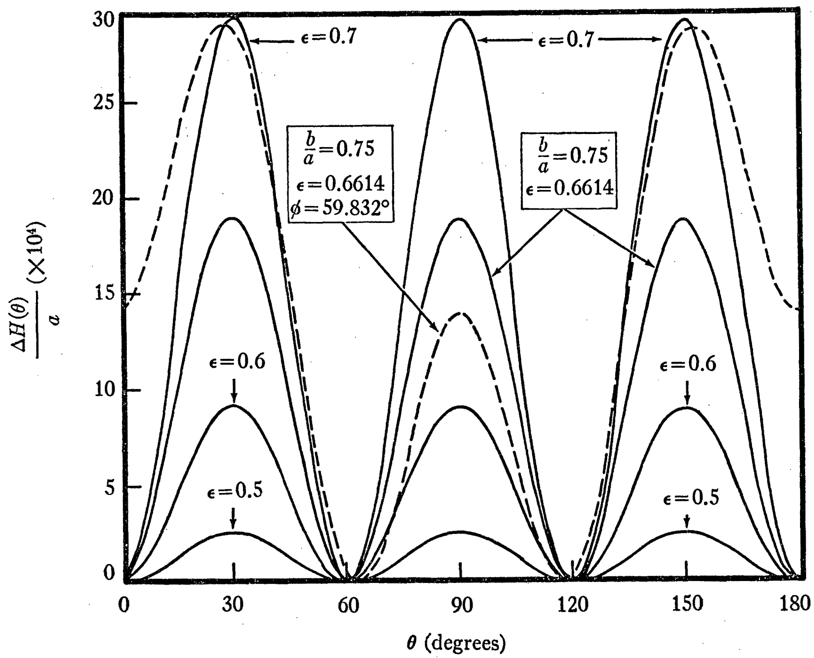


FIG. 3. Variation of $\Delta H(\theta)/a$ with θ for different ellipses. ($\phi=60°$ except for broken curve.)

value ($\phi = \phi'$, say) is plotted in Figure 2 versus the eccentricity of the ellipse, ϵ, defined by

$$\epsilon = (1 - b^2/a^2)^{1/2}. \tag{14}$$

When $\epsilon = 1$, $\phi' = 2 \cot^{-1} 2 = 53.10°$. For the ellipse in Figure 1, which has $b/a = 0.75$ ($\epsilon = 0.66144$), $\phi' = 59.832°$. For an ellipse this shape, $\Delta H + 1.42 \times 10^{-3} a$ is plotted (broken line) versus θ in Figure 3. The additional constant term insures that we are measuring the deviation from the smallest value of H (which does not occur at a simple angle) rather than the deviation from $H(0°)$. The pattern repeats with a period of 180°. If a = one inch, then the maximum variation in H would be 2.93 thousandths of an inch.

The value of ϕ which makes $\Delta H(90°) = 0$ is not the one which minimizes the greatest value of ΔH. In fact, the least variation in ΔH occurs for $\phi = 60°$, for which case $\Delta H(90°)$ is actually a maximum value of ΔH.

For this special case of an equilateral triangle we have

$$\begin{aligned} \Delta H(\theta) = &- [b + (b^2 + 3a^2)^{1/2}] + [(b^2 + m^2a^2)^{1/2} + \tfrac{1}{2}\{b^2(1 - \sqrt{3}m)^2 \\ &+ a^2(\sqrt{3} + m)^2\}^{1/2} + \tfrac{1}{2}\{b^2(1 + \sqrt{3}m)^2 + a^2(\sqrt{3} - m)^2\}^{1/2}]/(1 + m^2)^{1/2}. \end{aligned} \tag{15}$$

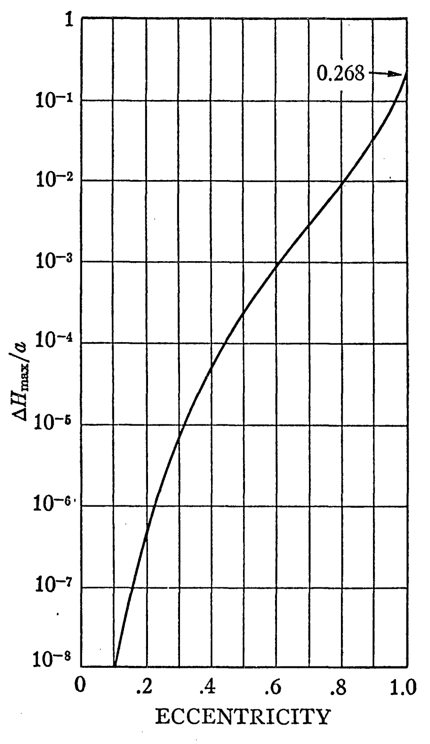


FIG. 4. Variation of $\Delta H_{max}/a$ with eccentricity for equilateral triangle case.

This function is plotted (solid curves) versus θ for different ellipses in Figure 3. The maximum value of $\Delta H(\theta)$, ΔH_{max}, is given by

$$\Delta H_{max} = a - b + (a^2 + 3b^2)^{1/2} - (b^2 + 3a^2)^{1/2}. \tag{16}$$

The variation of ΔH_{max} with eccentricity is plotted in Figure 4. Even on the log scale we lose information for very small eccentricities. For example, when $\epsilon = 0.02$, $\Delta H_{max}/a = 7.5 \times 10^{-13}$. When $\epsilon = 1$, $\Delta H_{max}/a$ is the difference between the hieght of an equilateral triangle of height 2 and the height ($\sqrt{3}$) of one whose side is 2.

For an ellipse the shape of that in Figure 1 with a = one inch, the maximum variation in the horizontal bar in a 60° V-groove measurement is 1.9 thousandths of an inch. It is therefore not surprising that in the experiment mentioned earlier, an ellipse the same shape as that in Figure 1 was observed to rotate inside the equilateral triangle.

To gain some insight into why the ellipse so nearly rotates, it is instructive to expand equation 16 as a power series in ϵ^2. We find that not only does the term in ϵ^2 disappear, but also that in ϵ^4. The first two nonzero terms are

$$\Delta H_{max} \approx 3a(4\epsilon^6 + 5\epsilon^8)/1024. \tag{17}$$

We have shown that any ellipse whose eccentricity is not too close to unity will, if measured in a 60° V-groove, give results essentially indistinguishable from those obtainable using a perfect circle. So, although one's first impressions might indicate that this type of measurement was suitable as a test for roundness, it clearly is not.

References

1. Michael Goldberg, Rotors in polygons and polyhedra, Mathematics of Computation, 14 (1960) 229–239.
2. Martin Gardner, Mathematical games, Scientific American, 208 (1963) 148–156.
3. Harry Shaw, Multipoint gauges, American Machinist, (1931) 43–46, 173–175 and 208–211.

A FACT ABOUT FALLING BODIES

WILLIAM C. WATERHOUSE, Cornell University

Suppose a body of mass m falls in a constant gravitational field g and encounters air resistance proportional to the nth power of its velocity v. Clearly v is governed by the equation

$$m\frac{dv}{dt} = mg - kv^n.$$

Calculus books often solve the equation for $n = 1$ or $n = 2$ and then point out that the velocity approaches a limit proportional to m or to $\sqrt{m}$.

It is actually easy to deduce this more generally. We have

$$\frac{dt}{dv} = \frac{1}{g[1 - (k/mg)v^n]},$$

so

$$t = \frac{1}{g}\int_0^v \frac{dv}{1-(k/mg)v^n} + t_0.$$

Since $1-(k/mg)v^n$ is a bounded factor times $1-(k/mg)^{1/n}v$, the integral diverges; that is, t gets arbitrarily large as v approaches $(mg/k)^{1/n}$. Hence that is the limiting velocity.

This uses nothing beyond the grasp of a calculus class, and is a good example of deriving information about a problem without knowing an explicit solution. It also furnishes what is all too often lacking, a nontrivial application of improper integrals.

TRIANGULAR MATRICES AND THE CAYLEY-HAMILTON THEOREM

GEORGE PHILLIP BARKER, University of Missouri, Kansas City

A well-known problem in elementary matrix theory texts states that for an $n\times n$ strictly upper triangular matrix S we have $S^n=0$. Here we give a generalization of this result and derive the Cayley-Hamilton theorem from it for matrices over certain fields.

Let us recall that an $n\times n$ matrix T is *upper triangular* (*strictly upper triangular*) if $t_{ij}=0$ for $i>j$ ($t_{ij}=0$ for $i\geqq j$). For the present, we shall make no restrictions on the field.

THEOREM. *If* $S_1, S_2, \cdots, S_n$ *are* $n\times n$ *triangular matrices such that the* (i, i) *element of* S_i *is zero, then*

$$S_1S_2\cdots S_n = 0.$$

Proof. Let $T=S_1S_2\cdots S_n$ and let

$$t_{ij} = \Sigma \cdots \Sigma s_{ii_1}s_{i_1i_2}\cdots s_{i_{n-1}j}.$$

(To avoid superscripts, the second subscript also indicates from which matrix the element comes. Thus $s_{i_pi_{p+1}}$ is the (i_p, i_{p+1}) element of S_{p+1}.)

We know that $t_{ij}=0$ unless $i<j$, since each S_p is triangular and t_{ii} is the product of the (i, i) elements of each S, one of which is zero.

We will be done if we can show that each summand

$$s_{ii_1}s_{i_1i_2}\cdots s_{i_{n-1}j}, \quad \text{where} \quad i\leqq i_1\leqq\cdots\leqq i_{n-1}\leqq j$$

is zero. To do this we consider the possible values of the indices. For this product to be nonzero it is necessary that no $s_{i_pi_{p+1}}$ be from the diagonal, that is, $i_p\neq i_{p+1}$. However, if $i=1$, then $i_1\geqq 2$. If $i_1=2$, then $i_2\geqq 3$; but if $i_1>2$, then $i_3\geqq 3$. In general, if $i_p=p+1$, then $i_{p+1}\geqq p+2$; but if $i_p>p+1$, then $i_{p+1}\geqq p+2$.

Thus

$$n \leqq i_{n-1} \leqq j \leqq n,$$

so $s_{i_{n-1}j} = s_{nn} = 0$. So we must have at least somewhere in the sequence $i_p = i_{p+1} = p+1$, and $s_{i_p i_{p+1}} = 0$.

On the other hand if $i > 1$, then $i_1 \geqq 2$, and the above argument holds. Thus $T = 0$.

COROLLARY. *If S is an $n \times n$ matrix in strictly upper triangular form, then $S^n = 0$.*

Now let us suppose that we have k matrices $M_1, \cdots, M_k$ each $n \times n$ and each partitioned the same way and each in block triangular form. If for $j = 1, 2, \cdots, k$,

$$M_j = \begin{bmatrix} T_1 & & & & & & \\ & \ddots & & & & * & \\ & & T_{j-1} & & & & \\ & & & S_j & & & \\ & & & & T_{j+1} & & \\ & 0 & & & & \ddots & \\ & & & & & & T_k \end{bmatrix}$$

with T_i triangular of order n_i and S_j is strictly upper triangular of order n_j, then

$$M_j^{n_j} = \begin{bmatrix} T_1' & & & & & & \\ & \ddots & & & & * & \\ & & T_{j-1}' & & & & \\ & & & 0 & & & \\ & & & & T_{j+1}' & & \\ & 0 & & & & \ddots & \\ & & & & & & T_k' \end{bmatrix}$$

Thus we have the

COROLLARY. *If the matrices $M_1, \cdots, M_k$ are as described above, then*

$$M_1^{n_1} M_2^{n_2} \cdots M_k^{n_k} = 0.$$

This is the partitioned form of the theorem.

We must now consider the field F from which come the entries of our matrices. In particular, we want a matrix A over F to be similar to a triangular matrix at least over the algebraic closure of F. This is certainly true if F is the real field, and we shall confine ourselves to matrices with real or complex entries. For such an A we let

$$p(\tau) = \det(\tau I - A)$$

be its characteristic polynomial.

Cayley-Hamilton Theorem. *For an $n \times n$ matrix A,*

$$p(A) = a_0 I + a_1 A + \cdots + A^n = 0.$$

Proof. Let $\lambda_1, \cdots, \lambda_k$ be the eigenvalues of A with multiplicities $n_1, \cdots, n_k$. Since

$$P^{-1}p(A)P = p(P^{-1}AP),$$

we may assume that A is triangular and, in fact, (cf. the proof of Schur's triangularization theorem, page 315 of [1]) that

$$A = \begin{bmatrix} \lambda_1 & & & & & & & & \\ & \ddots & & & & & & * & \\ & & \lambda_1 & & & & & & \\ & & & \lambda_2 & & & & & \\ & & & & \ddots & & & & \\ & 0 & & & & \lambda_2 & & & \\ & & & & & & \ddots & & \\ & & & & & & & & \lambda_k \end{bmatrix}$$

Now since

$$p(\tau) = (\tau - \lambda_1)^{n_1} \cdots (\tau - \lambda_k)^{n_k},$$

we have

$$p(A) = (A - \lambda_1 I)^{n_1} \cdots (A - \lambda_k I)^{n_k}.$$

But each $(A - \lambda_i I)^{n_1}$ is of the same form as the M_i of the second corollary above. Thus by that corollary

$$p(A) = 0.$$

Reference

1. Hans Schneider and G. P. Barker, Matrices and Linear Algebra, Holt, Rinehart and Winston, New York, 1968.

A GEOMETRIC PROOF OF THE FORMULA FOR ln 2

FRANK KOST, SUNY at Oneonta

Consider $f(x) = (1/x)$ defined on $[1, 2]$. The area under this curve and bounded by the lines $y=0$, $x=1$, $x=2$ is $\int_1^2 (1/x)\,dx = \ln 2$.

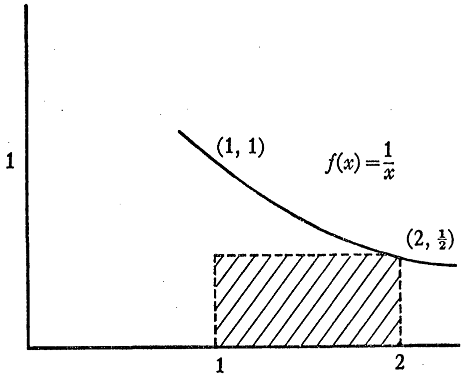


FIG. 1.

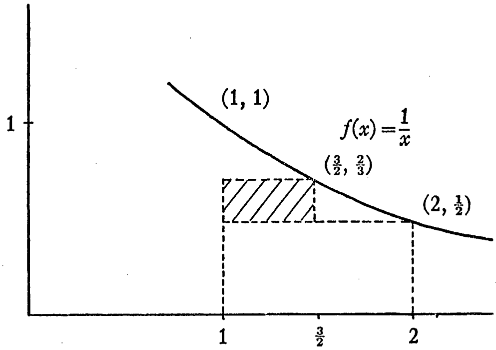


FIG. 2.

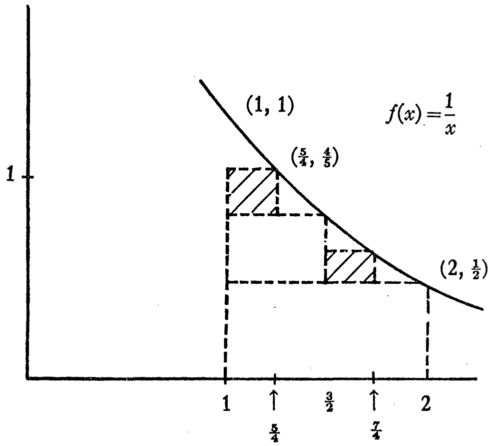


FIG. 3.

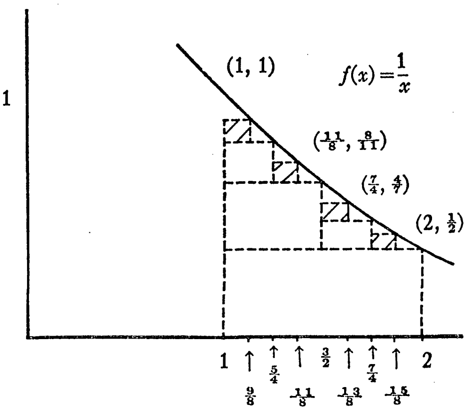


FIG. 4.

Our first approximation to this area is $A_1 = \frac{1}{2} = 1 - \frac{1}{2}$. See Figure 1. Next (Figure 2) we introduce the point $\frac{3}{2}$ and obtain a rectangle whose area $A_2 = [f(\frac{3}{2}) - f(2)]\frac{1}{2} = \frac{1}{3} - \frac{1}{4}$. In step 3 (Figure 3) we introduce the points $\frac{5}{4}$, $\frac{7}{4}$ and form two rectangles with total area

$$A_3 = \left[f\left(\frac{5}{4}\right) - f\left(\frac{3}{2}\right)\right]\frac{1}{4} + \left[f\left(\frac{7}{4}\right) - f(2)\right]\frac{1}{4} = \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8}.$$

In step 4 (Figure 4) add points $\frac{9}{8}$, $\frac{11}{8}$, $\frac{13}{8}$, $\frac{15}{8}$ and obtain four rectangles with combined area

$$A_4 = \left[f\left(\frac{9}{8}\right) - f\left(\frac{5}{4}\right)\right]\frac{1}{8} + \left[f\left(\frac{11}{8}\right) - f\left(\frac{3}{2}\right)\right]\frac{1}{8} + \left[f\left(\frac{13}{8}\right) - f\left(\frac{15}{8}\right)\right]\frac{1}{8} + \left[f\left(\frac{15}{8}\right) - f(2)\right]\frac{1}{8}$$
$$= \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \frac{1}{12} + \frac{1}{13} - \frac{1}{14} + \frac{1}{15} - \frac{1}{16}.$$

At the nth step ($n>1$) 2^{n-2} points and 2^{n-2} rectangles are introduced with combined area

$$A_n = \frac{1}{2^{n-1}} \sum_{k=2^{n-1}+1}^{2^n} f\left(\frac{k}{2^{n-1}}\right)(-1)^{k+1} = \sum_{k=2^{n-1}+1}^{2^n} \frac{(-1)^{k+1}}{k}.$$

The rectangles are disjoint and their union is the region under the curve. Hence $A_1+A_2+\cdots+A_n+\cdots = \sum_{k=1}^{\infty}(-1)^{k+1}/k = \ln 2$.

BOOK REVIEWS

EDITED BY D. ELIZABETH KENNEDY, University of Victoria

Materials intended for review should be sent to: Professor D. Elizabeth Kennedy, Department of Mathematics, University of Victoria, Victoria, British Columbia, Canada.

Reviews of texts at the freshman-sophomore level based upon classroom experience will be welcomed by the Book Review Editor.

A boldface capital **C** *in the margin indicates a classroom review.*

C *Elementary Linear Algebra.* By Bernard Kolman. The Macmillan Company, London, 1970. vi+255 pp.

This linear algebra book is intended for a one semester course for sophomores with the experience of a first course in calculus. This reviewer has found the book to be thoroughly thought out and well accepted by students.

All work is done over the real numbers. A brief discussion is given in the sixth chapter to working over the complexes.

The seven chapters are entitled: Preliminaries, Linear Equations and Matrices, Real Vector Spaces, Linear Transformations and Matrices, Determinants, Eigenvalues and Eigenvectors, and Differential Equations. The Preliminaries is a brief discussion of sets and functions. In the second chapter systems of equations, methods of solutions, matrices, algebraic properties of matrix operations, methods of finding the inverse of a matrix, row and column operations and equivalence of matrices are covered. The third chapter covers vector spaces, subspaces, linear independence, bases, dimension, isomorphisms, row and column spaces of a matrix and rank of a matrix. Chapter four deals with linear transformations between finite dimensional vector spaces. Topics covered are kernel, range, one-one, onto, matrix representation of a linear transformation, the vector space of all linear transformations between two finite dimensional vector spaces, and similarity of matrices. The fifth chapter gives an adequate treatment of determinants and applications. In the sixth chapter the problem of diagonalization of a linear transformation is taken up. Eigenvalues, eigenvectors, and characteristic polynomials are discussed. Kolman defines eigenspaces in an exercise. The reviewer found this section to be the only one in which some expansion seems desirable. A complete discussion of eigenspaces, dimension of eigenspaces and diagonalization is easily done and gives more complete results. The rest of this chapter deals with the standard inner product in R^n, Euclidean spaces, the Cauchy-Schwarz and triangle inequalities, orthogonality, Gram-Schmidt process, diagonalization of symmetric matrices and real quadratic forms. The seventh chapter deals with some applications of linear algebra to the solution of systems of linear differential equations.

W. E. PFAFFENBERGER, University of Victoria

The Theory of Numbers: An Introduction. By Anthony A. Gioia. Markham, Chicago, 1970. 185 pages. $8.50.

The book is designed for a first course in number theory which might occur anywhere from the third undergraduate year to the first graduate year. The background required is nominally only calculus, together with a limited amount of analytic geometry and linear algebra, but a first course in modern algebra, or at least a fair amount of mathematical sophistication, would be helpful. There are chapters on fundamental concepts, arithmetic functions, congruences and residues, the summatory arithmetic functions, sums of squares, continued fractions, the equation $x^n+y^n=z^n$ for $n\leqq 4$, an elementary proof of the prime number theorem, and the geometry of numbers. There is always, in the choice of topics to be covered by such a text, a fair amount of personal taste, as well as other considerations such as space and time, and with this in mind, the reviewer would like to have seen material on such topics as preprime number theorem, prime number theory (including some study of primes in arithmetic progressions), additive number theory, partitions and compositions, and additional material on irrational numbers.

The study of arithmetic functions is based on the use of the Dirichlet convolution, and this is rather well done. There are, however, certain notational problems, as follows. Unfortunately there does not seem to be a generally recognized notation for the operation of convolution. The author uses $f\cdot g$; other authors use $f\circ g$, $f\times g$, or $f*g$. The reviewer prefers a system where $f*g$ is used for Dirichlet convolution, and $f\cdot g$ or $f\circ g$ is reserved for the continuous generalization

$$f\circ g(x) = \sum_{n\leq x} f\left(\frac{x}{n}\right) g(n).$$

Also, the author uses f' for the inverse (under Dirichlet convolution) of f, rather than f^{-1}. The reviewer feels that f' should be reserved either for the ordinary derivative, or for the 'derivative' defined by $f'(n)=f(n)\log n$ (which, if $*$ is used as convolution, has the derivative-like properties $(f*g)'=f*g'+f'*g$, and $(f^{-1})'=-f'*(f*f)^{-1}$).

The chapter on continued fractions also includes material on Farey sequences and the Pell equation.

There are some 282 exercises, mostly of the 'prove that' variety (there are relatively few drill exercises, and no answers are provided). The level of difficulty varies from almost trivial to rather difficult; a student who does all will have a firm grasp of the material.

R. A. MACLEOD, University of Victoria

PROBLEMS AND SOLUTIONS

EDITED BY ROBERT E. HORTON, Los Angeles Valley College

Readers of this department are invited to submit for solution problems believed to be new that may arise in study, in research, or in extra-academic situations. Problems may be submitted from any branch of mathematics and ranging in subject content from that accessible to the talented high school student to problems challenging to the professional mathematician. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

The asterisk (*) will be placed by the problem number to indicate that the proposer did not supply a solution. Readers' solutions are solicited for all problems proposed. Proposers' solutions may not be "best possible" and solutions by others will be given preference.

Solutions should be legible and submitted on separate, signed sheets. Figures should be drawn in India ink and exactly the size desired for reproduction.

Send all communications for this department to Robert E. Horton, Los Angeles Valley College, 5800 Fulton Avenue, Van Nuys, California 91401.

Editor's Note: In order to provide more time for solvers to prepare solutions and thereby reduce the number of late solutions, the time between the proposal of problems and the deadline for submission of solutions is to be delayed one issue. Consequently, solutions to Problems 761 through 767 proposed in the May, 1970, issue, which normally would appear in this issue, will be delayed until the March, 1971, issue. The schedule hereafter will be as follows: Solutions to proposals appearing in the January issue must be mailed by the following July 15th. Solutions to proposals appearing in the March issue must be mailed by the following September 15th. Solutions to proposals appearing in the May issue must be mailed by the following November 15th. Solutions to proposals appearing in the September issue must be mailed by the following January 15th. Solutions to proposals appearing in the November issue must be mailed by the following May 15th. This new schedule will result in additional time for solvers of no less than six weeks.

To be considered for publication, solutions should be mailed before July 15, 1971.

PROPOSALS

779. Correction: *Proposed by Sidney H. L. Kung, Jacksonville University, Florida.*

Let $\alpha_1\ \alpha_2, \cdots, \alpha_n$ be real numbers. Show that $\Pi_{j=1}^{n} \left| \sin^j\alpha_j \cos\alpha_j \right| \leqq 1/(n+1)^{(n+1)/2}$, and if $0 \leqq \alpha_j \leqq \pi/2$ the equality sign holds if and only if

$$\alpha_j = \cos^{-1} 1/\sqrt{j+1}, \qquad j=1,2,\cdots,n.$$

782. *Proposed by Herta T. Freitag, Hollins, Virginia.*

1. Into an n-dimensional regular simplex of side a an n-dimensional hypersphere is inscribed. Inscribe another regular simplex into this hypersphere. Continue in this manner *ad infinitum.* Find:

a. the total contents of the entire set of simplexes.

b. the total contents of the entire set of hyperspheres.

2. Deal with the analogous problem concerning an n-dimensional hypercube.

783. *Proposed by H. L. Krall, University Park, Pennsylvania.*

1. If a_{ij} is the binomial coefficient $\binom{p}{q-i+j}$, evaluate d_n where

$$d_n = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n_1} & \cdots & a_{nn} \end{vmatrix}.$$

2. Show that $\sum_{k=0}^{n} \binom{n}{k} x_{\underline{k}|} y_{\underline{k}|} z_{\underline{n-k}|} (x+y+z-k)_{\underline{n-k}|} = (x+z)_{\underline{n}|} (y+z)_{\underline{n}|}$ where

$$x_{\underline{n}|} = x(x-1)\cdots(x-n+1).$$

784. *Proposed by Robert S. Doran, Texas Christian University.*

Let A be a ring with identity 1 without divisors of zero, and suppose $f:A \to A$ is a ring antihomomorphism such that $f(f(a)) = a$. Show that a and $f(a)$ commute whenever $af(-a) = 1$.

785. *Proposed by Michael Goldberg, Washington, D.C.*

In Problem 745, this MAGAZINE, solution May, 1970, we were asked: "Given any nine points in a unit square, show that among the triangles having vertices on the given points there exists at least one triangle whose area does not exceed 1/8." Closer bounds can be found.

a. In particular, find an arrangement of ten points such that the smallest triangle has an area of $(3\sqrt{17}-11)/32 = 0.0428$.

b. Find an arrangement of nine points in which the smallest triangle has an area greater than in a but less than 1/8.

786. *Proposed by J. A. H. Hunter, Toronto, Ontario, Canada.*

Let N be an integer with n digits, in the decenary system, such that the $(n-1)$th, $(n-2)$th, $(n-3)$th, $(n-4)$th, and $(n-5)$th digits are all 4's. What is the smallest N that is a perfect square?

787. *Proposed by T. J. Kaczynski, Lombard, Illinois.*

Suppose we have a supply of matches of unit length. Let there be given a square sheet of cardboard, n units on a side. Let the sheet be divided by lines into n^2 little squares. The problem is to place matches on the cardboard in such a way that: a) each match covers a side of one of the little squares, and b) each of the little squares has exactly two of its sides covered by matches. (Matches are not allowed to be placed on the edge of the cardboard.) For what values of n does the problem have a solution?

SOLUTIONS

Late Solutions

W. J. Blundon, Memorial University of Newfoundland: **754**; *Haig Bohigian, John Jay College of Criminal Justice, New York:* **754, 759**; *Richard L. Breisch, Pennsylvania State University:* **756**; *Ellis W. Detwiler, Adams, New York:* **754**; *Gerald C. Dodds, HRB-Singer, Inc., State College, Pennsylvania:* **757**; *M. G. Greening, University of New South Wales, Australia:* **759**; *Shiv Kumar, Panjabi University, Patiala, India, and Miss Nirmal, Government Girls' High School, Panipat, India (jointly):* **759**: *John Oman, Wisconsin State University at Oshkosh:* **754, 756**; *D. F. Paget, University of Tasmania, Australia:* **754**; *Kim R. Penrose, Billings, Montana:* **744**; *Simeon Reich, Israel Institute of Technology, Haifa, Israel:* **757**; *Phil Tracy:* **745**; *C. S. Venkataraman, Trichur, South India:* **754, 758**; *Michael Yoder, Albuquerque, New Mexico:* **755, 756, 757.**

Comment on Problem 657

657. [May, 1967, and January, 1968] *Proposed by C. Stanley Ogilvy.*

Ship A is anchored 9 miles out from a point O on a straight shoreline. Ship B is anchored 3 miles out opposite a point 6 miles from O. A boat is to proceed from A to some point on the shore, pick up a passenger, and take him to ship B. It costs the boat owner $1 per mile to run his boat, whether there is a passenger aboard or not. Where should the owner contract to pick up the passenger so that his net profit (from A to shore to B) shall be a maximum? We can assume that the passenger insists on a straight line course from the pickup point to B.

Comments by Charles W. Trigg, San Diego, California.

The proposer's comment on page 44 of the January, 1968, issue that there is no pickup point for a maximum net profit is based upon the assumption that the boat owner will *charge* the passenger a *fixed amount per mile*. There is nothing in the problem as stated to warrant this assumption. The problem merely states, "It costs the boat owner $1 per mile to run his boat." The assumption made also assumes the passenger to be a prime sucker, even though he is smart enough to insist "on a straight line course from the pickup point to B." It would hardly seem that he would fall for a fixed rate per mile. Water taxi mileage meters are as likely to be undependable as those in land taxis.

A more reasonable assumption would be that the passenger had normal intelligence and contracted to pay a *fixed fee* for the *entire pickup trip*. In that case the minimum distance would yield the maximum profit, as concluded in the "ten incorrect solutions."

There must be a better way to emphasize that a zero of the first derivative does not always indicate a maximum, and that a maximum does not always exist.

Comment on Problem 700

700. [September, 1968, and March, 1969] *Proposed by Charles W. Trigg.*

Find a prime number which is the sum of primes in two ways such that either set of addends and the sum together contain no duplicated digits.

Comment by the proposer.

The number of odd addends must be odd. There are five odd digits, so there must be exactly *three* odd addends. There are four possible terminal digit situations: $1+3+5=9$, $1+5+7={}^{1}3$, $3+5+9={}^{1}7$, and $5+7+9={}^{2}1$. Since 1 is not a prime it can appear only as a terminal digit. Also, 5 may not appear as a terminal digit of a larger prime.

2 may appear alone as an addend and not as a terminal digit. In this case, there may be *two* or *four* addends. The possible terminal digit situations are: $2+1=3$, $2+3=5$, $2+5=7$, $2+7=9$, $2+9={}^{1}1$, $2+1+5+9={}^{1}7$, $2+3+7+9={}^{2}1$, and $2+5+7+9={}^{2}3$.

If any prime addend had four digits, the other addend would have to be 2, and duplicate digits would appear in the sum. Thus the search is reduced to considering primes of three digits or less with no duplicate digits. There are 120 of these.

The three solutions to the problem are:

$$5+19+23=47=5+29+13$$
$$5+29+367=401=5+7+389$$
$$5+43+761=809=5+61+743$$

The only other primes which can be expressed as the sum of primes without duplication of digits are:

$$5=2+3,\quad 7=2+5,\quad 41=5+7+29,\quad 61=2+59,$$
$$67=5+19+43,\quad 89=5+23+61,\quad \text{and}$$
$$103=2+5+7+89.$$

Comment on Problem 728

728. [May, 1969, and January, 1970] *Proposed by G. L. N. Rao, J. C. College, Jamshedpur, India.*

Find the sum of the infinite series:

$$\frac{x-2}{x^2-x+1}+\frac{2x^2-4}{x^4-x^2+1}+\frac{4x^4-8}{x^8-x^4+1}+\cdots$$

when $|x|>1$.

Comment by Shiv Kumar, Panjabi University, Patiala, India, and Miss Nirmal, Government Girls' High school, Panipat, India.

The problem can be solved as follows:

(1) $$\frac{x+2}{x^2+x+1}-\frac{x-2}{x^2-x-1}\equiv\frac{2(x^2+2)}{x^4+2x^2+1}$$

(2) $$\frac{x^2+2}{x^4+x^2+1}-\frac{x^2-2}{x^4-x^2+1}\equiv\frac{2(x^4+2)}{x^8+x^4+1}$$

$$(3)\qquad \frac{x^4+2}{x^8+x^4+1}-\frac{x^4-2}{x^8-x^4+1}\equiv\frac{2(x^8+2)}{x^{16}+x^8+1}$$

In general writing powers of x as $x^2, x^{2^2}, x^{2^3}, \cdots$ we have

$$(n+1)\qquad \frac{x^{2^n}+2}{x^{2^{n+1}}+x^{2n}+1}-\frac{x^{2^n}-2}{x^{2^{n+1}}-x^{2n}+1}=\frac{2(2^{2^{n+1}}+2)}{x^{2^{n+2}}+x^{2^{n+1}}+1}$$

Multiplying equation (2) by 2, equation (3) by $2^2, \cdots$ equation $(n+1)$ by 2^n and adding, this results in

$$\frac{x+2}{x^2+x+1}-S_{n+1}\equiv\frac{2(2^{2^{n+1}}+2)}{2^{2^{n+2}}+2^{2^{n+1}}+1}$$

where S_{n+1} represents the sum of $n+1$ terms of the given series. As $n\to\infty$ the expression on the right hand side approaches zero if $|x|>1$. Thus

$$S_{n+1}=\frac{x+2}{x^2+x+1}$$

for $|x|>1$ and diverges for $|x|\leqq 1$.

Comment on Problem 743

743. [November, 1969, and May, 1970] *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

Let P be an interior point of a regular tetrahedron, $T\equiv A_1A_2A_3A_4$, with $p_i=PA_i$, and let x_{ij} denote the distance of P from the edge A_iA_j. Then prove

$$\sum_{i=1}^{4} p_i \geqq 2\sqrt{3}/3 \sum_{i<j} x_{ij},$$

equality holding if and only if P is at the center O of T.

Comment by E. F. Schmeichel, College of Wooster, Ohio.

The inequality should read

$$\sum_{i=1}^{4} p_i \geqq \frac{2\sqrt{2}}{3}\sum_{i<j} x_{ij}.$$

Apparently a $\sqrt{3}$ was misprinted in place of the $\sqrt{2}$ above. To show that the inequality as printed is false consider a regular tetrahedron of edge length 1. Let P be the midpoint of edge A_1A_2. Then $p_1=p_2=1/2$, $p_3=p_4=\sqrt{3}/2$, $x_{12}=0$, $x_{13}=x_{14}=x_{23}=x_{24}=\sqrt{3}/4$ and $x_{34}=\sqrt{2}/2$.

Thus

$$\sum_i p_i = 1+\sqrt{3} < 2.8 \quad\text{and}\quad \sum_{i<j} x_{ij} = \sqrt{3}+\sqrt{2}/2$$

Since $\sqrt{6}>2.4$ we have

$$\frac{2\sqrt{3}}{3}\sum_{i<j} x_{ij} = 2+\sqrt{6}/3 > 2.8$$

so for the point in question

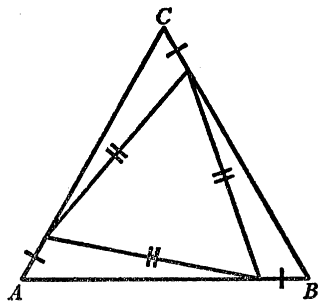
$$\sum_i p_i < \frac{2\sqrt{3}}{3}\sum_{i<j} x_{ij}.$$

By the continuity of the distances involved, this inequality will be retained for interior points of the tetrahedron sufficiently near P.

Comment on Problem 754

754. [March and November, 1970] *Proposed by N.S.F. Class at University of California, Berkeley.*

Show that the triangle ABC is equilateral.



Comments by J. F. Rigby, University College, Cardiff, Wales.

These comments are the results of my attempts to solve Problem 754 in this MAGAZINE:

In Figure 3, show that the triangle ABC is equilateral.

The problem can be generalized in various ways; the vertices of the equilateral triangle XYZ can lie on the sides of the triangle ABC produced (Theorem 1), we can consider regular n-gon inscribed in an n-gon (Theorem 2), and we can work in the hyperbolic plane as well as in the Euclidean plane. The basic results of hyperbolic geometry and trigonometry used in the article can be found in any textbook of non-Euclidean geometry. We use the notation "$[LMN]$" to mean "M lies between L and N on the line LN."

DEFINITIONS. *Let $ABCD\cdots$ be an n-gon in the Euclidean or hyperbolic plane, and let $U, V, W\cdots$ be points on the lines $AB, BC, CD\cdots$ such that $AU=BV=CW=\cdots$. If $[AUB]$, $[BVC]$ $[CWD]$, $\cdots$ then the n-gon $UVW\cdots$ is regularly inscribed in the first way in $ABCD\cdots$* (eg., see Figure 13).

If $[ABU]$, $[BCV]$, $[CDW]$, $\cdots$ *then* $UVW \cdots$ *is regularly inscribed in the second way* (e.g., see Figure 11). *If* $[UAB]$, $[VBC]$, $[WCD]$, $\cdots$ *then* $UVW \cdots$ *is regularly inscribed in the third way* (e.g., see Figure 12).

THEOREM 1. *Let* ZXY *be an equilateral triangle regularly inscribed in the triangle* ABC *in any of the three ways. Then* ABC *is equilateral.*

To prove this we need four lemmas, all of which are true in both the Euclidean and the hyperbolic planes.

LEMMA 1. *If a triangle contains an obtuse angle or a right angle, then the side opposite that angle is greater than the other two sides.*

LEMMA 2. *The angles of an equilateral triangle are acute.*

LEMMA 3. *If the triangles* XYZ *and* PQR*, situated as in Figure* 1 *or Figure* 2, *are equilateral, then* $PZ > SZ$.

Proof. $\theta > \phi = \psi = \omega$.

Note. In the proof of Theorem 1 it may happen that we obtain a figure similar to Figure 1, in which P, Q, R coincide but the lines XRQ, YPR, ZQR are still equally inclined to each other. We shall use the phrase "PQR is equilateral" in this case also; Lemma 3 is still true in this case.

LEMMA 4. *If, in the triangle* ZPS *(as in Figure* 1 *for example),* $PZ > SZ$ *and* A *is a point between* P *and* S*, then* $PZ > AZ$.

Proof of Theorem 1. (i) Suppose ZXY is regularly inscribed in the first way (Figure 4), so that $AZ = BX = CY$. Assume without loss of generality that $\alpha \leqq \beta$, $\alpha \leqq \gamma$. Construct angles α', α'' as shown, equal to α. The two new lines so constructed, and the line BXC, form an equilateral triangle. (It is not immediately obvious that the lines ZQ and XB meet, or that ZP and YP meet, but certainly YR meets XC; then the triangles ZPY and XQZ are congruent to YRX.) Then $BX \leqq QX = RY \leqq CY$ by Lemmas 1 and 2; but $BX = CY$ so we must have equality everywhere, which implies $\alpha = \beta = \gamma$. Hence ABC is equilateral.

(ii) Suppose ZXY is regularly inscribed in the second way (Figures 5–8). Without loss of generality, either (a) $\alpha \geqq \beta \geqq \gamma$ or (b) $\alpha \leqq \beta \leqq \gamma$.

(a) (Figure 5 or Figure 6) Construct angles β', β'' as shown, equal to β, to form the equilateral triangle PQR. Then $CY \leqq RY = PZ \leqq AZ$ by Lemmas 1 and 2. We proceed as in case (i).

(b) (Figure 7 or Figure 8) Construct angles β', β'' as shown, equal to β, to form the equilateral triangle PQR. Then $CY \geqq RY = PZ \geqq AZ$ by Lemmas 3 and 4. We proceed as in case (i).

(iii) The final case, when ZXY is regularly inscribed in the third way, is similar to case (ii).

This completes the proof.

We shall show that certain generalizations of the original problem are not possible, by constructing various counterexamples.

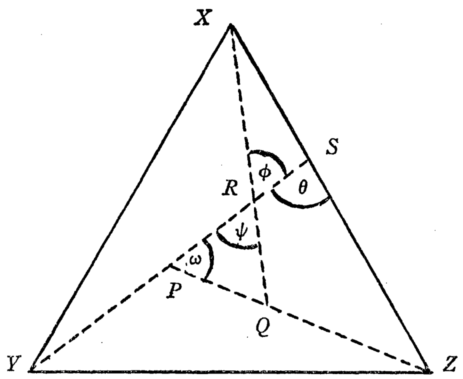


FIG. 1.

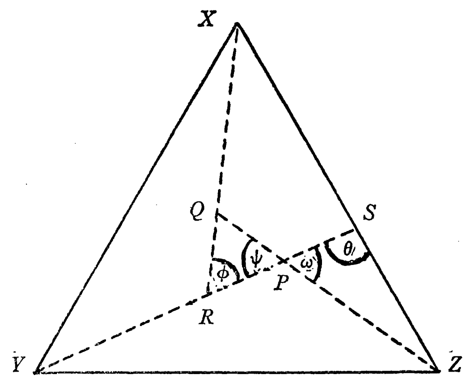


FIG. 2.

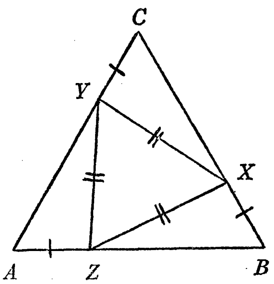


FIG. 3.

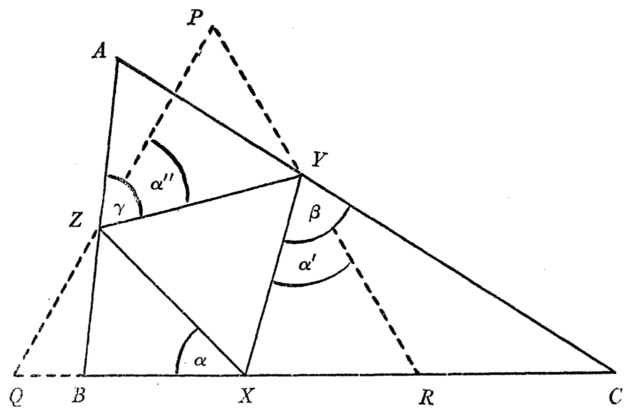


FIG. 4.

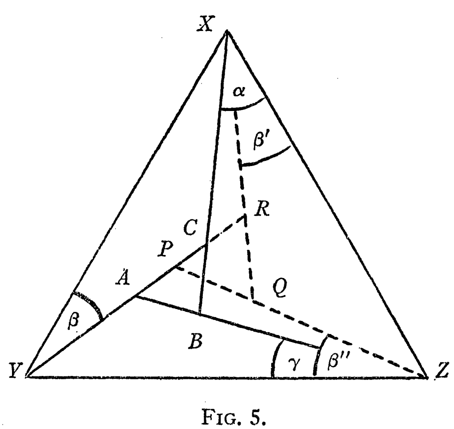


FIG. 5.

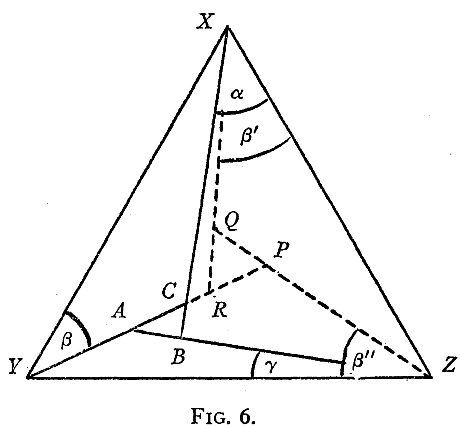


FIG. 6.

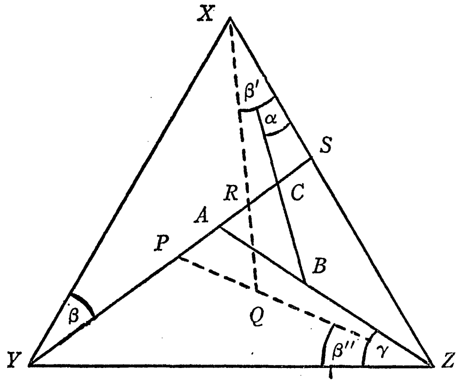


FIG. 7.

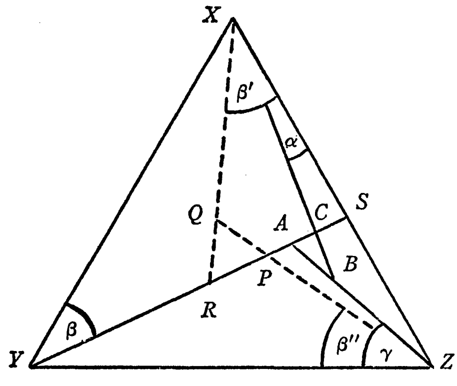


FIG. 8.

The half-line with B as endpoint containing C is called the *ray* BC. If, instead of requiring that ZXY be regularly inscribed in ABC, we require only the weaker condition that Z lies on the ray AB, X on the ray BC, and Y on the ray CA, then ABC need not be equilateral, as is shown by the counterexample below (Figure 9).

COUNTEREXAMPLE 1. *In both the Euclidean and the hyperbolic planes, there exist nonequilateral triangles* ABC, *and points,* X, Y, Z *such that*

(i) $[BXC]$, $[CYA]$ *and* $[ABZ]$,
(ii) $BX=CY=AZ$,
(iii) XYZ *is equilateral.*

Proof. In Figure 9, let XYZ be an equilateral triangle, and L any point on XY produced; then $XZ<XL$. If $\angle XZL$ is acute, let M be the point between L and Z such that $XM=XZ$; if $\angle XZL$ is obtuse or right, let $M=Z$.

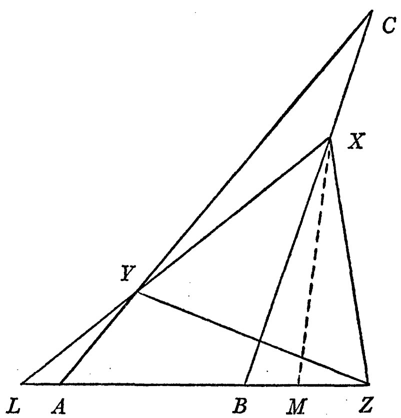


FIG. 9.

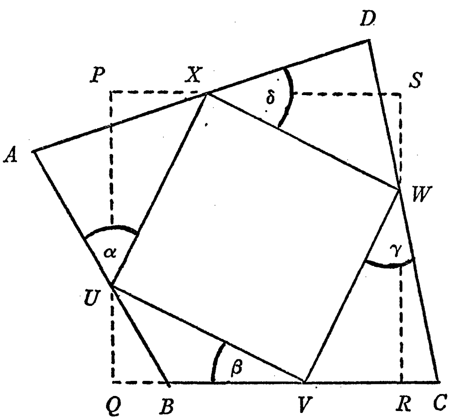


FIG. 10.

Let B be a variable point on the closed interval LM. If B is distinct from M, then $BX>MX=ZX=XY$; hence there exists a unique point C on BX produced such that $BX=CY$. Let CY meet LM at A; the positions of C and A depend on B. If $B=M$, we define $C=X$ (so that $BX=CY$) and $A=L$.

Write $f(LB)=BX-AZ$; then $f(LB)$ is a continuous function of LB. When B is at L we have $f(0)=LX-LZ>0$, while $f(LM)=MX-LZ=ZX-LZ<0$. Hence there exists B between L and M such that $f(LB)=0$. For this position of B we have $AZ=BX=CY$ but ABC is not equilateral.

Let $ABCD\ldots$ be a plane convex n-gon, and let the regular n-gon $UVW\ldots$ be regularly inscribed in $ABCD\ldots$. We can show that $ABCD\ldots$ is regular in the cases covered by the following theorem:

THEOREM 2. (a) *Suppose that $UVW\ldots$ is regularly inscribed in the first way. If $n=4$ and the plane is Euclidean or hyperbolic, or if $n>4$ and the plane is hyperbolic and $UVW\ldots$ is so large that its angles are acute or right, then $ABCD\ldots$ is regular.*

(b) *Suppose that $UVW\ldots$ is regularly inscribed in the second or third way. If $n\geqq 4$ and the plane is Euclidean, or if $n>4$ and the plane is hyperbolic and $UVW\ldots$ is so small that its angles are obtuse or right, then $ABCD\ldots$ is regular.*

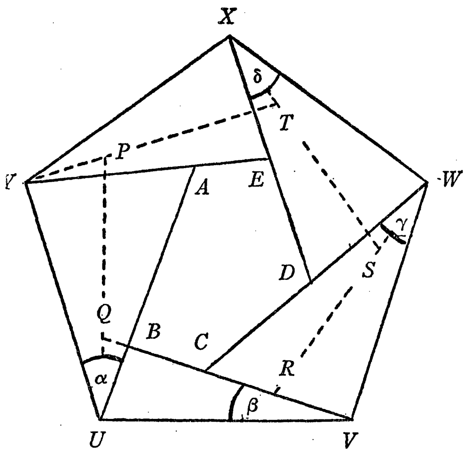


FIG. 11.

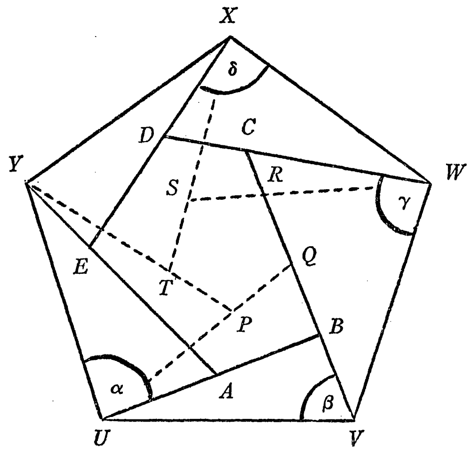


FIG. 12.

Outline of proof (Figures 10, 11, 12). The method of proof is the same as in the proof of Theorem 1. To construct the regular n-gon $PQR\ldots$ (corresponding to the equilateral triangle PQR in the proof of Theorem 1) we make use of the smallest of the angles $\alpha, \beta, \gamma, \delta, \ldots$ In the figures it is assumed without loss of generality that β is the smallest of these angles; the figures show $UVW\ldots$ inscribed in $ABCD\ldots$ in the first, second and third ways respectively.

In Figure 10, $PQR\ldots$ has acute or right angles (in the hyperbolic case because it lies outside $UVW\ldots$ and hence has smaller angles than does $UVW\ldots$). In Figures 11 and 12, $PQR\ldots$ has obtuse or right angles (in the hyperbolic case because it lies outside $UVW\ldots$ and therefore has larger angles than does $UVW\ldots$). Hence in each figure the angle CRW is obtuse or right. Hence $BV\leqq QV=RW\leqq CW$; but $BC=CW$ so we must have equality everywhere. Thus $Q=B$, $R=C$, and $\alpha=\beta=\gamma$. We can now prove that $\beta=\gamma=\delta$, etc. This completes the proof.

COUNTEREXAMPLE 2. *In Figure* 13, $UVWXYZ$ *is a regular hexagon in the Euclidean plane, the triangles* AUZ, CWV, EYX *are congruent with angles of* 100°, 30°, 50°, *and* BVU, DXW, FZY *are congruent with angles of* 140°, 10°, 30°. *We can show by elementary trigonometry that* $AU=BV= \dots$ *but* $ABCDEF$ *is not regular.*

This is a simple case of Counterexample 4 at the end of the paper, which shows that Theorem 2(a) cannot be extended to Euclidean $2n$-gons ($n\geqq 3$) or to small hyperbolic $2n$-gons ($n\geqq 3$).

The relevance of the next lemma will become apparent in the counterexample that follows it.

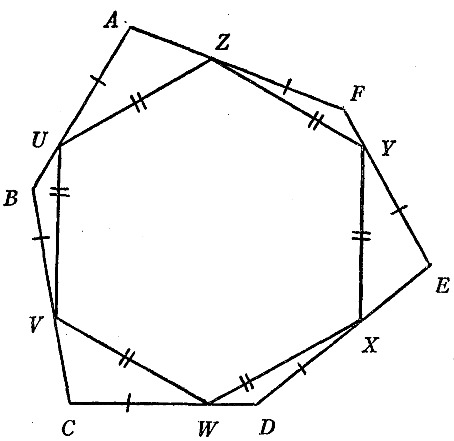


FIG. 13.

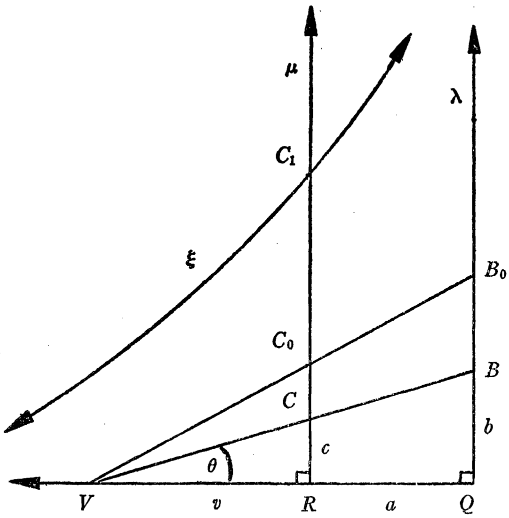


FIG. 14.

LEMMA 5. *Let* RQ *be a line-segment in the hyperbolic plane* (*Figure* 14) *and let* λ, μ *be rays through* Q, R *perpendicular to* RQ *on the same side of* RQ. *Then there exist points* B *on* λ *and* V *on* QR *produced such that* $RV>QB$ *and* $RC=VB-VQ$ (*where* C *is the meet of* BV *and* μ).

Proof. Let the common parallel ξ to the rays λ or QR meet μ at C_1. Regard V as a variable point of QR produced, and let B_0 be the point of λ such that $QB_0=RV=v$, say (using the standard hyperbolic unit of length). Let $RQ=a$, and write $VB_0=v+a+x_0$ (where $x_0>0$ since $VB_0>VQ$). Then, from the right-angled triangle VQB_0,

$$\begin{aligned}\cosh(v+a)\cosh v &= \cosh(v+a+x_0)\\ &= \cosh(v+a)\cosh x_0+\sinh(v+a)\sinh x_0.\end{aligned}$$

Hence

$$\begin{aligned}\cosh v &= \cosh x_0+\tanh(v+a)\sinh x_0\\ &< \cosh x_0+\sinh x_0 = e^{x_0}.\end{aligned}$$

Hence $x_0 \to \infty$ as $v \to \infty$.

Choose a particular value of v such that $x_0 > RC_1$; V and B_0 are now to be regarded as "fixed," and

$$\text{(i)} \qquad VB_0 - VQ = x_0 > RC_1 > RC_0.$$

Let B be a variable point on the closed interval QB_0, and write $QB = b$, $RC = c$, $VB = v + a + x$ $(x > 0)$, $f(b) = VB - VQ - RC = x - c$. Then from (i)

$$\text{(ii)} \qquad f(QB_0) > 0.$$

From the right-angled triangles BQV, CRV we have

$$\sec\theta = \tanh VB/\tanh VQ = \tanh(v + a + x)/\tanh(v + a),$$

$$\tan\theta = \tanh c/\sinh v.$$

Hence

$$\frac{\operatorname{sech}^2(v + a + x)}{\tanh(v + a)} \frac{dx}{d\theta} = \sec\theta\tan\theta$$

and

$$\frac{\operatorname{sech}^2 c}{\sinh v} \frac{dc}{d\theta} = \sec^2\theta\cdot$$

Hence, when $\theta = 0$, $dx/dc = 0$. Hence $x < c$ for all sufficiently small positive values of x; i.e., $f(b) < 0$ for all sufficiently small values of b.

Since $f(b)$ is a continuous function of b, we deduce (using (ii) also) that there exists B between Q and B_0 such that $f(b) = 0$, i.e., such that $RC = VB - VQ$.

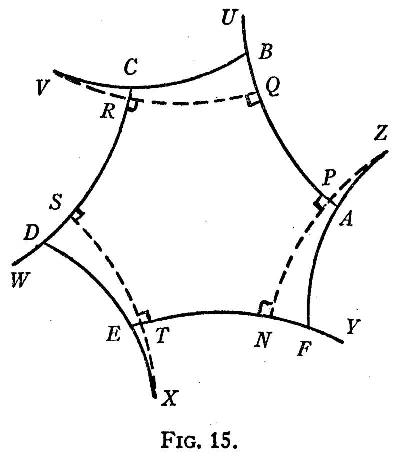


FIG. 15.

COUNTEREXAMPLE 3. *Let $PQR \ldots$ be a regular $2n$-gon in the hyperbolic plane, with $n \geqq 3$, all of whose angles are right-angles. (It is well known that such $2n$-gons exist.) The case $n = 3$ is illustrated in Figure 15 by the hexagon $PQRSTN$, using the angle-preserving Poincaré model of the hyperbolic plane. Let V and B be points* *determined as in Lemma* 5, *such that* $RV > QB$ *and* $RC = VB - VQ$. *Let* U, W, X . . . *be points as shown in the figure such that* $QU = RV = SW = TX$ . . . *and let* D, F, . . . *be points such that* $QB = SD = NF =$ . . . *Then*

$$VB = VQ + RC = WR + RC = WC,$$

and so by congruent triangles we have $AU = BV = CW = DX =$ . . . *Hence* UVW . . . *is regular, and is regularly inscribed in* $ABCD$ . . . *in the second way, but* $ABCD$ . . . *is not regular.*

This proof of the existence of counterexamples fails when $n = 2$, because we have used the existence of right angles in the figure to make the hyperbolic trigonometry more manageable; there is no regular 4-gon with right angles in the hyperbolic plane. However, it is not difficult to extend Lemma 5 to the case where $\angle RQB$ is obtuse and $\angle RQB = \angle QRC$, so that we can construct similar counterexamples starting with any acute-angled regular $2n$-gon PQR . . . . This covers the case $n = 2$ also. Thus in the hyperbolic plane we cannot extend Theorem 2 to *all* regular $2n$-gons UVW . . . regularly inscribed in the second way.

Note that Theorem 2(b) is proved for all sufficiently small regular polygons UVW . . . in the hyperbolic plane (namely those with obtuse or right angles), whilst counterexample 3 is liable to produce rather large polygons UVW . . . Thus the question "under what circumstances can a given regular $2n$-gon in the hyperbolic plane be regularly inscribed in the second way in a nonregular $2n$-gon?" is not completely answered. A similar remark applies in the hyperbolic plane to Theorem 2(a) and counterexample 4 below.

COUNTEREXAMPLE 4. *When* $n \geqq 3$, *any regular* $2n$*-gon in the Euclidean plane, and any sufficiently small* $2n$*-gon in the hyperbolic plane, can be regularly inscribed in the first way in a nonregular* $2n$*-gon.*

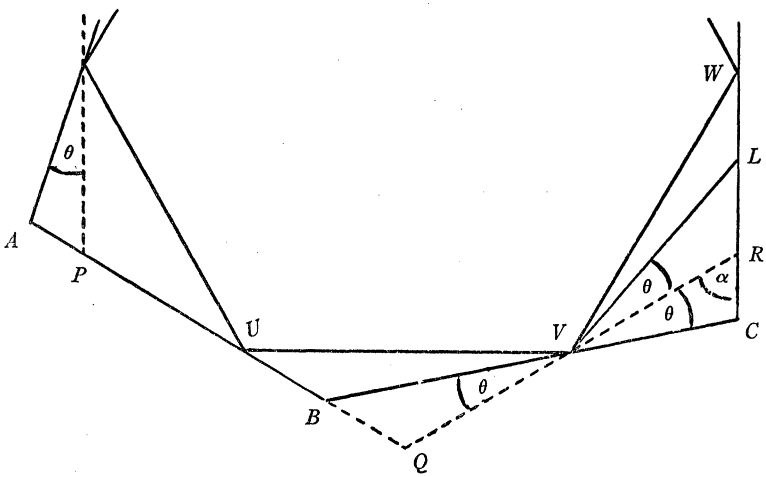


FIG. 16.

Outline of proof. Let PQR . . . be a regular $2n$-gon, and let UVW . . . be the regular $2n$-gon whose vertices are the midpoints of PQ, QR, . . . (Figure 16). We shall show that, under certain conditions, there exists $\theta(0<\theta<\angle RVW)$ such that $BV=CW$, i.e., $LV=CW$. Then ABC . . . will be a nonregular $2n$-gon in which UVW . . . is regularly inscribed in the first way.

Regard θ as a variable angle, so that C, L are variable points. Then

$$\frac{LV-RV}{CW-RV}=\frac{LV-RV}{CW-RW}=\frac{LV-RV}{RL}\,\frac{RL}{CR}\to(\cos\alpha)1\quad\text{as }\theta\to 0.$$

Now $\cos\alpha<1$; hence $LV<CW$ for all sufficiently small positive values of θ. However, when $\theta=\angle RVW$, $LV>CW$ under certain conditions. These conditions are clearly satisfied for all regular Euclidean $2n$-gons when $n\geqq 3$ (but not when $n=2$), and for all sufficiently small regular $2n$-gons ($n\geqq 3$) in the hyperbolic plane. Now $LV-CW$ is a continuous function of θ; hence there exists θ $(0<\theta<\angle RVW)$ such that $LV=CW$.

There are other similar ways of showing the existence of more general counterexamples for $2n$-gons inscribed in the first way. Counterexample 2 was obtained by showing that the relevant value of θ in that particular case is 20°; this shows that we cannot in general expect to be able to construct counterexamples using straight edge and compasses.

Various interesting questions remain, of which I will mention three: (a) Can we find counterexamples for $(2n+1)$-gons when $n\geqq 2$? (b) Is Theorem 2(b) *always* true in the hyperbolic plane for regular polygons inscribed in the third way, or are there counterexamples for sufficiently large polygons? (c) Can we extend counterexample 4 (using a different method) to show that every hyperbolic $2n$-gon ($n\geqq 3$) with obtuse angles can be regularly inscribed in the first way in a nonregular $2n$-gon?

Comment on Q359

Q359. [May, 1965] Minimize $\int_0^1 F'(x)^2dx$ where $F(0)=0$ and $F(1)=1$.

[*Submitted by Murray S. Klamkin*]

Comment by Sidney Spital, California State College at Hayward.

An alternative solution is obtained by letting $G(x)=F(x)-x$. Then clearly since $G(0)=G(1)=0$, we have

$$\int_0^1 (G'(x)+1)^2dx=\int_0^1 (G'(x))^2dx+1\geqq 1$$

Comment on Q465

Q465. [November, 1969] Prove that the product of any n consecutive positive integers is divisible by $n!$

[*Submitted by E. F. Schmeichel*]

Comment by Hugh M. Edgar, San Jose State College, California.

If m is any positive integer then

$$n! \mid m(m+1)\cdots(m+n-1),\ n!(m-1)! \mid (m-1)!m(m+1)\cdots(m+n-1) = (m+n-1)! \quad\text{and}\quad \frac{(m+n-1)!}{n!(m-1)!} \in z$$

are equivalent statements and the last statement is true because

$$\frac{(m+n-1)!}{n!(m-1)!} = \binom{m+n-1}{n}$$

is a binomial coefficient.

Comment on Q472

Q472. [March, 1970] Let $\alpha \geqq 0$, $\beta \geqq 0$, and $\alpha+\beta \leqq \pi/2$. Prove

$$2 + \tan\alpha + \tan\beta \leqq 2\sec\alpha + 2\sec\beta.$$

[*Submitted by Harley Flanders*]

Comment by Cecil G. Phipps, Tennessee Technological University.

The printed answer is not as quick as it could be with the given conditions on α and β. Consider the sum of the four inequalities: $\tan\alpha < \sec\alpha$, $\tan\beta < \sec\beta$, $1 \leqq \sec\alpha$, $1 \leqq \sec\beta$. The answer follows at once with the inequality sign holding at all times.

Comment on Q473

Q473. [March, 1970] Let d be a metric which generates the usual topology in the real line. Does it follow that $d(0, x) \neq d(0, y)$ if $0 < x < y$ are real numbers?

[*Submitted by E. F. Schmeichel*]

Comment by Albert Wilansky, Lehigh University.

It is easy to get a stronger result. Draw an arbitrary simple arc in the plane (nonclosed and without endpoints). It is homomorphic with R and so its (Euclidean) metric can be transferred to R. By making the curve horseshoe shaped we ensure that the metric for R has $d(0, x) > d(0, y)$ for some $0 < x < y$. In fact, for large x, $d(0, x)$ is monotonely decreasing!

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q492. Let $f(z)$ be regular in the whole plane except on the segment $z = yi$, $-1 \leqq y \leqq +1$. Suppose $f(z)$ is real for $z = x > 0$. Is $f(z)$ necessarily real for $z = x < 0$?

[*Submitted by Harley Flanders*]

Q493. A number of points are given inside a triangle. Connect these points as well as the vertices of the triangle by segments that do not cross each other, until the interior of the triangle is covered with small triangular regions. Show that the number of triangular regions is always odd.

[*Submitted by Loring Tu*]

Q494. In a given sphere APB, CPD and EPF are three mutually perpendicular and concurrent chords. If $AP=2a$, $BP=2b$, $CP=2c$, $DP=2d$, $EP=2e$ and $FP=2f$, determine the radius of the sphere.

[*Submitted by Murray S. Klamkin*]

Q495. If the squares of the cotangents of a series of angles are in harmonic progression, show that the squares of the cosines of the same angles are in harmonic progression.

[*Submitted by Charles W. Trigg*]

Q496. Find two linearly independent functions whose Wronskian vanishes identically.

[*Submitted by C. Stanley Ogilvy*]

Q497. Determine the condition for concurrency of the three straight lines,

$$a_ix + b_iy + c_i = 0,$$

$i=1, 2, 3.$

[*Submitted by Charles W. Trigg*]

Q498. Given a cake in the form of a triangular layer (prism) which is covered with a thin coat of icing on its top and sides. Show how to divide the cake into eleven portions so that each portion contains the same amount of cake and icing.

[*Submitted by Murray S. Klamkin*]

Q499. Every open set U in the real line E' is the union of countable disjoint open intervals.

[*Submitted by Warren Page*]

Q500. Show that $4x^3+6x^2+4x+1$ is composite for all positive integral values of x.

[*Submitted by Norman Schaumberger*]

Q501. Prove that $\prod_{i=1} \cos(r_i\pi/15) = (1/2)^7$.

[*Submitted by C. S. Venkataraman*]

Q502. If f is differentiable on $[0, 1]$, $f(0)=0$ and $a>0$, prove there exists a number c in $(0, 1)$ such that

$$f'(c) = \frac{ac^{a-1}f(c)}{1-c^a}\cdot$$

[*Submitted by Erwin Just*]

Q503. A boy walks 4 mph, a girl walks 3 mph, and a dog walks 10 mph. They all start together at a certain place on a straight road, and the boy and girl walk steadily in the same direction. The dog walks back and forth between the two of them, going repeatedly from one to the other and back again. After one hour where is the dog and which direction is he facing?

[*Submitted by A. K. Austin, University of Sheffield*]

Q504. Find the error: The Cantor set is defined by the use of the rational numbers 1/3, 2/3, 1/9, 2/9, 7/9, 8/9, $\cdots$. Since the Cantor set consists of this subset of the rationals, the Cantor set is countably infinite.

[*Submitted by Francis Siwiec*]

Q505. Solve the differential equation $(x-a)(x-b)y''+2(2x-a-b)y'+2y=0$.

[*Submitted by Gregory Wulczyn*]

Q506. If in the plane a convex polygon P is contained in an arbitrary polygon Q $(Q\neq P)$, then the perimeter of P is less than that of Q. Give a proof whose method is readily applicable to prove an extension of the statement to higher dimensions.

[*Submitted by Hwa S. Hahn*]

Q507. For what values of N is $7N+55$ a factor of N^2-71?

[*Submitted by David L. Silverman*]

Q508. A shipping clerk wants to package a 30-inch diameter sphere in a cubical box measuring 32 inches on a side, using eight identical small spheres in the corners of the box to prevent any movement of the large sphere. What must the diameter of the small spheres be?

[*Submitted by Harold B. Curtis*]

Q509. Evaluate $\lim_{n\to\infty} \sum_{k=0}^{n} (k^2+3k+1)/(k+2)!$.

[*Submitted by Erwin Just and Helen Jick*]

Q510. $1/1+1/2+1/3+1/6=2$. Find two integers other than 6 which also have the property that the sum of the reciprocals of their divisors equals 2.

[*Submitted by David L. Silverman*]

Q511. Define $E(x)=\sum_{n=0}^{\infty} x^n/n!$. Show that $E(x)E(y)=E(x+y)$ without the obvious identification of $E(x)$.

[*Submitted by E. F. Pinzka*]

Q512. Find the solution of $x^{(x+1)}+x^x-1=0$.

[*Submitted by Patricia La Fratta*]

(Answers on pages 16–19.)

MATHEMATICS MAGAZINE, Vol. 44, No. 1, JANUARY-FEBRUARY, 1971